Extension Complexity of Matroid Polytopes*

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Abstract

Given a matroid, or any polytope, its extension complexity was characterised Yannakakis [Yan88]. Examining extension complexity of polytopes gives insight into time complexity of solving optimisation problems using LP based methods. In this survey, we examine the extension complexity of some matroid polytopes.

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Contents

1 Introduction

The extension complexity of a polytope P is the smallest number of facets among convex polytopes Q that can have P as a projection. In this case, Q is called an extended formulation of P and it may have a higher dimension than P. The extension complexity of a polytope is a natural question to ask since many problems can be encoded as a combination of linear constraints. If this admits a small extended formulation, then standard techniques like linear program solvers can be used to solve the problem. In 1980s, Swart [Swa87] attempted to prove that there is a polytope with only polynomial many facets that projects down to the TSP polytope. Note that a polynomial size extended formulation for the TSP polytope would imply $P = NP$. The purported linear programs were extremely complicated to analyze. However, in a breakthrough paper, Yannakakis [Yan88] refuted all such attempts by showing that every symmetric linear program (LP) for the TSP polytope has exponential size. Since the proposed LP by Swart was symmetric, that refuted Swart's proof.

However, the question of extension complexity of non-symmetric LPs remained open. However, in 2010, Kaibel, Pashkovich, and Theis [KPT10] gave examples of polytopes that do not have polynomial size symmetric extended formulations, but that do have polynomial size asymmetric extended formulations. This rekindled interest in the lingering question. In another breakthrough paper in 2012, Fiorini, Masser, Pokutta, Tiwary, and de Wolf [FMP+12] finally proved that the TSP polytope does not admit any polynomial size extended formulation, symmetric or not.

In this survey we examine many matroid polytopes and give an overview of known extension complexities and open questions. We begin with k-l sparsity matroids, which can be viewed as dilworth truncation of spanning trees which is known to have a small extension as shown in [IKK+16]. Then, we examine extension complexity from the lens of hitting sets and communication complexity ([Apr22], [FFGT15], [KLdW15]). Hitting set of polytopes is quite interesting and can be possibly used as an alternative method for small extension of regular matroids. As a first step, we see the effect of 2-sums on size of hitting set using [ACF18]. Moreover, we examine effect of this on extension of regular matroids.

Moreover, we examine the extension complexity of polygons as shown in [FRT12]. A related question was raised on Open Problem garden, which asks if there exist infinitely many n such that a convex polygon on n vertices has extension complexity $\Omega(n)$. This was answered by Shitov [Shi20], who showed that every n-gon is a projection of polytope with atmost $147 \cdot n^{2/3}$ facets. Finally, we examine the extension complexity of exact bipartite matching polytope which was shown to be exponential by Svennson et. al. [JSY].

2 Preliminaries

Matroid theory originated in the middle of the 1930s. There is a huge literature on matroids by now. For an introduction, see for example the excellent textbooks of Oxley [Oxl06] or Schrijver [S+03].

Moreover, our problem also deals with the extension complexity of matroids, for which refer the excellent lecture notes by Li [Li18] as an introduction. Below we give some basic definitions and facts about matroids and extension complexity of polytopes.

2.1 Matroid

A matroid M is a pair $M = (E, I)$, where E is the finite ground set and $I \subseteq P(E)$ is a nonempty family of subsets of E that satisfies the following two axioms.

1. Closure under subsets. For every $I \in \mathcal{I}$ and $J \subseteq I$ we have $J \in \mathcal{I}$.
2. Augmentation property. For every $I, J \in \mathcal{I}$ where $|I| < |J|$, there is an $j \in J$ such that $I \cup j \in \mathcal{I}$.

We denote $m = |E|$ throughout the paper. The sets in I are called the independent sets of M. An inclusion-wise maximal set $B \in \mathcal{I}$ is called a base. Note that by the augmentation property, all base sets have the same size. Let $\mathcal{B} \in \mathcal{I}$ denote the collection of base sets.

Definition 2.1 (Dual Matroid)**.** The dual of a matroid M is another matroid M^* that has the same elements as M, and in which a set is independent if and only if M has a basis set disjoint from it. Alternatively, basis sets of dual matroid are complements of basis sets of M.

Definition 2.2 (Circuits)**.** For a matroid M, any inclusion-wise minimal dependent subset is called a circuit.

Similarly, we can define the notion of co-circuits as circuits in the dual matroid.

2.1.1 Matroid Rank

. Motivated by Linear Algebra, there is a rank-function of a matroid that is defined for every subset $A \subseteq E$ as the size of the largest independent set that is contained in A,

$$rank(A) = max\{|I| \mid I \in \mathcal{I} \text{ and } I \subseteq A\}$$

The size of every maximal independent set is rank(E). This number is called the rank of M. The matroid problem is to compute a maximal independent set.

An important property of the rank-function is its submodularity. In general, a function $f : P(E) \rightarrow \mathbb{R}$ is called submodular, if for any sets $S, T \subseteq E$, we have

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$$

.

2.1.2 Matroid Polytope

The polytopes we consider in this paper are convex polytopes defined as the convex hull of finitely many points in $\mathbb{R}^m$. Any convex polytope P can be described as the intersection of halfspaces, i.e., as $P = \{x \in \mathbb{R}^m | Ax \leq b\}$, for some matrix $A \in \mathbb{R}^{k \times m}$ and vector $b \in \mathbb{R}^k$. A face of the polytope P is the set of points in P minimizing or maximizing a linear function. If the polytope P is described by $Ax \leq b$, then any face of P can be described as $\{x \in P \mid A'x \leq b'\}$, where $(A'\ b')$ is some subset of the rows of $(A\ b)$. With every matroid, there is an associated matroid polytope. This polytope is crucial for our arguments.
For a set $I \in E$, its characteristic vector $x^I \in R^E$ is defined as

$$x = \begin{cases} 1, & \text{if } e \in I \\ 0, & \text{otherwise} \end{cases}$$

For any collection of sets $A \subseteq P(E)$, the polytope $P(A) \subset R^E$ is defined as the convex hull of the characteristic vectors of the sets in A,

$$P(A) = conv\{x^I \mid I \in A\}$$

For a matroid $M = (E, I)$, its matroid polytope is defined as $P(I) \subset \mathbb{R}^E$, i.e., the convex hull of the characteristic vectors of the independent sets. The points $\{x^I \mid I \in \mathcal{I}\}$ are the corners of the matroid polytope $P(I)$. Edmonds [Edm70] gave a simple description of this polytope which uses the rank function of the matroid (see also [Sch03]). For convenience, we define for any $x \in \mathbb{R}^E$ and $S \subseteq E$,

$$x(S) = \sum_{e \in S} x_e$$

Lemma 2.3 ([Edm03])**.** *For a matroid (E, I) with rank function r, a point $x \in \mathbb{R}^E$ is in $P(I)$ iff*

$$x_e \geq 0 \;\forall\; e \in E(1) \tag{1}$$

$$x(S) \leq r(S) \;\forall\; S \subseteq E \tag{2}$$

It is easy to see that any 0-1 corner of the polytope given by (1) and (2) corresponds to an independent set in $\mathcal{I}$. The nontrivial part is to show that the described polytope does not have a non-integral corner. Let $\mathcal{B}$ be the family of base sets of the matroid $(E, \mathcal{I})$. Let n be the rank of the matroid, i.e., the size of any base set. The matroid base polytope, defined as $P(B)$, is clearly a face of the matroid polytope $P(I)$. Putting the following equation together with (1) and (2) will give a description of $P(B)$,

$$x(E) = n \tag{3}$$

2.1.3 Matroid Intersection

The matroid intersection problem is, given two matroids $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$ over the same ground set, compute a maximum size set in $I_1 \cap I_2$, the common independent sets. Note that in general $(E, I_1 \cap I_2)$ is not a matroid anymore.

2.1.4 Matroid Intersection Polytope

The intersection of two matroids also has an easy polytope description: Edmonds [Edm70] showed a surprising result that one can describe the matroid intersection polytope $P(I_1 \cap I_2)$ just by putting together the constraints of the two matroid polytopes $P(I_1)$ and $P(I_2)$ (see also [S$^+$03])

Theorem 2.4 ([Edm03])**.** *For two matroids (E, I_1) and (E, I_2),*

$$\mathcal{P}(M_1 \cap M_2) = \mathcal{P}(M_1) \cap \mathcal{P}(M_2)$$

That is, a point $x \in \mathbb{R}^E$ is in the polytope $\mathcal{P}(M_1 \cap M_2)$ iff

$$x_e \geq 0 \ \forall \ e \in E \tag{4}$$

$$x(S) \leq r_1(S) \ \forall \ S \subseteq E \tag{5}$$

$$x(S) \leq r_2(S) \ \forall \ S \subseteq E, \tag{6}$$

where r_1 and r_2 are the rank functions of the two matroids, respectively.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the families of base sets of the matroids $(E, I1)$ and $(E, I2)$, respectively. Note that there can be a common base set only if the two matroids have same rank, say n. To obtain the common base polytope $P(B_1 \cap B_2)$ one just needs to put the constraint 3 together with inequalities 4, 5 and 6.

2.1.5 Chaining tight sets

In this section, we state a lemma (see [S$^+$03]) which expresses any tight constraint as a linear combination of tight constraints obtained by a chain of sets.

Definition 2.5 (Tight sets)**.** Let $\mathcal{M}$ be a matroid with the rank function r. Let $\boldsymbol{x} \in \mathcal{P}(\mathcal{M})$ (matroid polytope). We call a set S a tight set of $\boldsymbol{x}$ with respect to r if $x(S) = r(S)$

Lemma 2.6 (Uncrossing operation)**.** *Let $\mathcal{M}$ be a matroid with rank function r, and let $\boldsymbol{x} \in \mathcal{P}(\mathcal{M})$. If S and T are tight sets of $\boldsymbol{x}$ with respect to r, then so are $S \cup T$ and $S \cap T$.*

Lemma 2.7 (Maximal Chain of Tight Sets)**.** *Let $\mathcal{M}$ be a matroid with rank function r, and let $\boldsymbol{x} \in \mathcal{P}(\mathcal{M})$. Let $\mathcal{C} = \{C_1, \cdots, C_k\}$ with $\phi \subset C_1 \subset \cdots \subset C_k$ be an inclusion-wise maximal chain of tight sets of $\boldsymbol{x}$ with respect to r. Then every tight set T of $\boldsymbol{x}$ with respect to r must satisfy $\chi_T \in span\{\chi_C : C \in \mathcal{C}\}$*

2.1.6 Partition Matroid

Partition matroid is a matroid in which E is partitioned into (disjoint) sets $E_1, E_2, .., E_l$ and

$$I = \{X \subseteq E : |X \cap E_i| \ \leq k_i \ \forall \ i = 1, .., l\},$$

for some given parameters $k_1, .., k_l$

2.1.7 Partition Matroid Polytope

For a given partition matroid $P = (E, \mathcal{I})$ and partition $E_1, E_2, .., E_l$, a point $x \in R^E$ is in the polytope $P(I_1 \cap I_2)$ iff

$$x_e \geq 0 \; \forall \; e \in E \tag{7}$$

$$x(E_i) \leq r_1(E_i) \; \forall \; i \tag{8}$$

2.1.8 Matroid Union

Matroid union of $M_1, M_2, M_3, .., M_k$ is defined as

$$M = M_1 \vee M_2 \vee .. \vee M_k = (\cup_{i=1}^{k} S_i, \mathcal{I} = \{\cup_{i=1}^{k} I_i | I_i \in \mathcal{I}\})$$

We can show that M is a matroid, and derive the rank function for it which is given by,

$$r_m(U) = \min_{T \subseteq U} \left[U \backslash T + \sum_{i=1}^{k} r_{M_i}(T \cap S_i) \right]$$

2.2 Extension Complexity

Given a polytope $P \subseteq \mathbb{R}^n$ as a convex hull of finitely many points (called vertices) in $\mathbb{R}^n$, by Minkowski- Weyl theorem, it is equivalent to a bounded polyhedron Q that is described by a system of linear equalities. Let $|Q|$ denote the number of linear inequalities in Q's description. There might be different LP formulations for the same polytope. Now, LP solvers can give a solution to an optimization problem on this polytope if the LP formulation is polynomial in $|Q|$. Hence, there are two directions to investigate, as given in [Li18].

1. Find another description Q' with polynomially many inequalities;
2. Find a higher dimension polytope (bounded polyhedron) $H \subseteq \mathbb{R}^k$ (where $k > n$) such that it projects to P, and H has a $poly(n)$ description.

In some cases, it can be shown that 1 is not possible by showing that any description in $\mathbb{R}^n$ has exponential size, hence we can only turn to 2. Note that any optmization problem over P can be done by optimizing the same objective function over H.

Definition 2.8. Let $P \subseteq \mathbb{R}^n$ be a polytope, a polytope $H \subseteq \mathbb{R}^k$ in a higher dimensional space is called an extended formulation of P if $\pi(H) = P$, where $\pi : \mathbb{R}^k \to \mathbb{R}^n$ is the projection map. The extension complexity of P denoted by $xc(P)$, is defined to be :

$$xc(P) := \min_{Q:\ Q \text{ is an extended formulation of } P} |Q|$$

We say P has a compact formulation if $xc(P) = poly(n)$.

Now, linear programs with compact formulations as defined above can be solved in PTIME by using an LP solver on the extended formulation of the LP. Note that if a polytope does not have a compact formulation, that does not imply that the problem cannot be solved in PTIME, one of the most famous examples being the matching polytope. Finding a maximum matching for a general graph has a polynomial time algorithm, but it has been shown recently that the matching polytope has no compact formulation [Rot17].

2.2.1 Non-negative rank characterization

A characterization of the extension complexity was given by [Yan88] in terms of the non-negative rank of the "slack matrix" associated with the polytope.

Definition 2.9 (Slack matrix)**.** Given a polytope $P \subseteq \mathbb{R}^n$ suppose it is described by $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ where $A_{m\times n}$ is a matrix and $b \in \mathbb{R}^m$ is a vector. Let the set of vertices of P be $\{z_1, z_2, \cdots, z_k\}$. The slack matrix of P under the above description is defined to be the matrix

$$S_{m\times k} := (b - Az_1, b - Az_2, \cdots, b - Az_k) \in \mathbb{R}^{m\times k}_{\geq 0}$$

Definition 2.10 (Non-negative rank)**.** Given a matrix $S_{m\times k}$, its non-negative rank, denoted by $\text{rank}_+(S)$, is defined to be the least integer $r \in \mathbb{N}$, such that $S = L_{m\times r}R_{r\times k}$ where L and R are two non-negative matrices.

Now, with these definitions, we state Yannakakis's characterization of the extension complexity of a polytope.

Theorem 2.11 ([Yan88])**.** *For any polytope $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ whose dimension ≥ 1, let S be its slack matrix, one has*

$$xc(P) = rank_+(S)$$

3 Results

In this section, we examine the existing literature on the extension complexity of k-l sparsity matroids, regular matroids, polygons and methods of evaluating extension complexity such as hitting set, communication complexity.

3.1 Dilworth truncation of matroids

It is interesting to examine the extension complexity of matroids obtained upon some operation on a matroid with small extension. One of such operations is truncation of a matroid. In this section, we see the definition of truncation of a matroid and show that for transversal matroids, truncation has a small extension complexity via flows, as shown in [HIH21]. As a corollary

3.1.1 Definitions

Definition 3.1 (Set function)**.** A set function $\rho : 2^E \to \mathbb{R}$ is a β_0-function if it satisfies the following:

1. $\rho(X) \geq 0$ for $\phi \neq X \subseteq E$
2. $\rho(Y) \leq \rho(X)$ for $Y \subseteq X \subseteq E$ (monotonicity)
3. $\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$ for $X, Y \subseteq E$ (submodularity)

Definition 3.2 (Polymatroid)**.** Given a set function $\rho : 2^E \to \mathbb{R}$, a polytope

$$P(\rho) = \{x \in \mathbb{R}^E : x \geq 0, x(X) \leq \rho(X)(\phi \neq X \subseteq E)\}$$

is a polymatroid where $x(X) = \sum_{e \in X} x_e$.

Definition 3.3 (l-lower truncated polymatroid)**.** For this polymatroid $P(\rho)$, consider l satisfying $0 \leq l \leq \min\{\rho(\{e\}) | e \in E\}$. Then, $\rho' : 2^E \to \mathbb{R}$ defined by $\rho'(X) = \rho(X) - l$ ($X \subseteq E$), (E, ρ') is a β_0-function, which defines a polymatroid $P(\rho')$. This polymatroid is called an l-lower-truncated polymatroid obtained from (E, ρ), simply a lower-truncated polymatroid.

3.1.2 Lower truncated tranversal and $k - l$ sparsity polymatroid

In this section, we shall see the connection of extension complexity of lower truncated transversal matroid with network flow.
However, first recall the notion of transversal matroid.

Definition 3.4 (Transversal Matroid)**.** Let $G = (U, W; A)$ be a bipartite graph with left vertex set U,right vertex set W and directed edge set $A \subseteq U \times W$ where edges are directed from U to W. Define $\Gamma(X)$ to be $\{w | (u, w) \in A, u \in X\}$. Define $(U, \mathcal{I})$ as the transversal matroid where $\mathcal{I}$ contains $X \subseteq U$ that can be covered by a matching.

By Hall's theorem, $X \subseteq U$ can be covered by a matching iff $|Y| \leq \Gamma(Y)$ for all $Y \subseteq X$.

Definition 3.5 (Lower truncated transversal polymatroid)**.** For $G = (U, W; A)$,let k, l be positive integers with $d'k - l > 0$, where d' is the minimum degree of a vertex in U. Define $\gamma_{k,l} : 2^U \to \mathbb{R}$ by $\gamma_{k,l}(X) = k\gamma_N(X) - l$ ($X \subseteq U$). $(U, \gamma_{k,l})$ is a polymatroid, called a (k, l)-lower-truncated transversal polymatroid.

First, we establish relation between the $k - l$ transversal polymatroid and $k - l$ sparsity matroid. Consider a case of $|\Gamma(\{u\})| = 2(u \in U)$ in the bipartite graph G. Let $G' = (V, E)$ be an undirected graph with vertex set $V = W$ and edge set $E = U$. G' can be regarded as a graph obtained from G by subdividing each edge e of G' by a vertex e of G. (see fig. 1)

Then, we define :

Definition 3.6 ($k - l$ sparsity matroid)**.** For G' and positive integers k, l with $2k - l > 0$, $\{X : |Y| \leq k|V(X)| - l(\phi \neq Y \subseteq X)\}$ is the set of independent sets of a matroid, called (k, l)-sparsity matroid or count matroid, where $V(X)$ denotes the set of vertices on which edges in X are incident.

Observe that $M(G; k, l)$ is isomorphic to $k - l$ sparsity matroid, since $V(X)$ in G' is the same as $\Gamma(X)$ in G for $X \subseteq E$. So, we only need to examine $M(G; k, l)$ now.

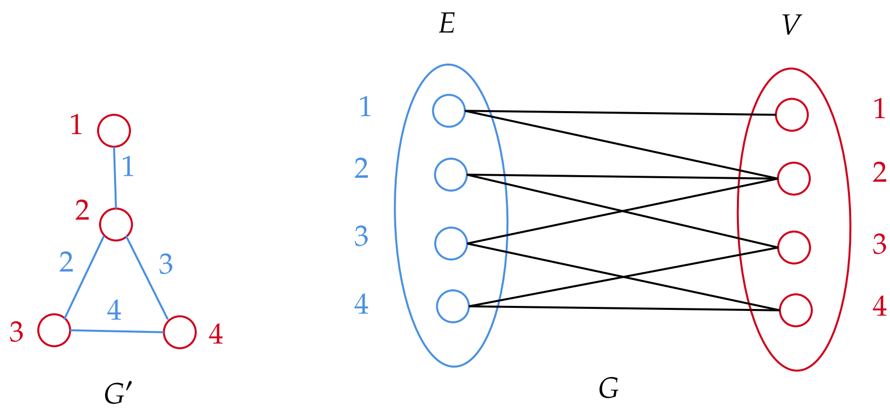


Figure 1: Relating G and G' for an example graph

3.1.3 Extension complexity of lower truncated transversal polymatroid

In this section, we examine the extension complexity of $M(G; k, l)$. Recall that given the polymatroid $P(\rho)$, consider l satisfying $0 \leq l \leq \min\{\rho(\{e\}) | e \in E\}$. Let $\rho' : 2^E \to \mathbb{R}$ be defined by $\rho'(X) = \rho(X) - l$ $(X \subseteq E)$. Let us define χ_e $(e \in E)$ to be the unit vector on the underlying set E with $\chi_e(e) = 1$ and others 0.

Lemma 3.7. *$x \in P(\rho')$ iff $x + l\chi_e \in P(\rho)$ for each $e \in E$.*

Proof. Suppose $x \in P(\rho')$. For each non empty $X \subseteq E$,we have $x(X) \leq \rho'(X) = \rho(X) - l$,and hence $(x + l\chi_e)(X) \leq x(X) + l \leq \rho(X)$, which implies $x + l\chi_e \in P(\rho)$.
Conversely, if $x + l\chi_e \in P(\rho)$ for $e \in E$, then, for X with $e \in X \subseteq E$, we have $x(X) + l \leq \rho(X)$. This holds for each $e \in E$, and the lemma follows. □

Theorem 3.8. *$xc(P(\rho')) \leq |E| \cdot xc(P(\rho))$.*

The proof of 3.8 follows easily, since we add constraints corresponding to $x + l\chi_e$ for all $e \in E$ using 3.7, hence we have $|E| \cdot xc(P(\rho))$ constraints. In order to examine the transversal polymatroid, we examine the network flow polytope.

Definition 3.9 (Network flow)**.** Let $\tilde{N} = (\tilde{V}, \tilde{A}; S, t; \tilde{c})$ be a network with a vertex set $\tilde{V}$, a directed edge set $\tilde{A}$,a set of sources $S \subset \tilde{V}$, a unique sink $t \in \tilde{V} - S$, and a capacity $\tilde{c} \in \mathbb{R}^{\tilde{A}}$ where $\tilde{c}(a)(\geq 0)$ is a capacity of $a \in \tilde{A}$. For $f \in \mathbb{R}^{\tilde{A}}$, we define $\delta_+ f \in \mathbb{R}$ by $\delta_+ f(v) = -\sum_{e=(u,v)\in\tilde{A}} f(e) + \sum_{e=(v,w)\in\tilde{A}} f(e)$ for $v \in \tilde{V}$. The restriction of $\delta_+ f$ to a subdomain $X(\subseteq \tilde{V})$ is denoted by $\delta_+ f|_X$. $f \in \mathbb{R}^{\tilde{A}}$ is a flow if $0 \leq f(a) \leq \tilde{c}(a)(a \in \tilde{A})$, $\delta_+ f|_{V-(S\cup\{t\})} = 0$ and $\delta_+ f|_S \geq 0$.

Definition 3.10 (Cut function)**.** A cut function $\tilde{c} : 2^{\tilde{V}-\{t\}} \to \mathbb{R}$ is defined by $\tilde{c}(Y) = \sum_{e=(u,w)\in\tilde{A}, u\in Y, w\in\tilde{V}-Y} \tilde{c}(e)$ for $Y \subseteq \tilde{V} - \{t\}$.

Definition 3.11 (Network flow polymatroid)**.** Define the function $\gamma_{\tilde{N}} : 2^S \to \mathbb{R}$ as $\gamma_{\tilde{N}}(X) = \min\{\tilde{c}(Y) | Y \subseteq \tilde{V} - \{t\}, Y \cap S = X\}$ for $X \subseteq S$. Then, $(S, \gamma_{\tilde{N}})$ is a polymatroid.

With this definition, we have the following well-known result.

Lemma 3.12. *For $x \in \mathbb{R}^S$, $x \in P(\gamma_{\tilde{N}})$ iff there is a flow f with $x = \delta_+ f|_S$. Hence $xc(P(\gamma_{\tilde{N}})) \le 2|\tilde{A}| + |S|$.*

Now, we create a flow network to get the extension complexity of transversal polymatroid. Consider the bipartite graph $G = (U, W; A)$ with left vertex set U,right vertex set W and directed edge set $A \subseteq U \times W$ where edges are directed from U to W, with an extra vertex t and directed edges from W to t. Let $c(e) = \infty$ for $e \in A$, $c(e) = 1$ for (w, t). Let this network be N, then we have $\gamma_N(X) = \Gamma(X)$. So, as a consequence we get

Lemma 3.13. $xc(P(\gamma_N)) = |A| + |W|$. *For the transversal matroid over U of bipartite graph G, its independence polytope has extension complexity of $|A| + |U| + |W|$.*

Moreover, using 3.13 and 3.8 we get

Theorem 3.14. $xc(P(\gamma_{k,l})) \le |U|(|A| + |W|)$.

However, we can also get an extension directly using the network flow idea. For the network $N = (U \cup W \cup \{t\}, A'; U, t; c)$, we replace c with a new capacity c_w for $w \in W$ defined by $c_w(e) = +\infty$ $(e \in A)$, $c_w(e) = k$ $(e = (w', t), w' \in W - \{w\})$, and $c_w(e) = k - l$ $(e = (w, t))$.

Lemma 3.15. *For $x \in \mathbb{R}^U$, $x \in P(\gamma_k, l)$ iff there is a flow f with $x = \delta_+ f|_S$ in network $N_w = (U \cup W \cup \{t\}, A'; U, t; c_w)$ for each $w \in W$.*

Using this lemma when $k \ge l$, we get a better bound,

Theorem 3.16. *When $k \ge l$, $xc(P(\gamma_{k,l})) \le |W|(|A| + |W|)$.*

Applying these results for $k - l$ sparsity matroids we get

Theorem 3.17. *For the independence polytope of a (k, l)-sparsity matroid of a graph $G' = (V, E)$, there is an extended formulation of size $|E|(2|E| + |V| + 1)$ in general and that of size $|E| + |V|(2|E| + |V|)$ when $k \ge l$.*

3.2 Relating communication and extension complexity

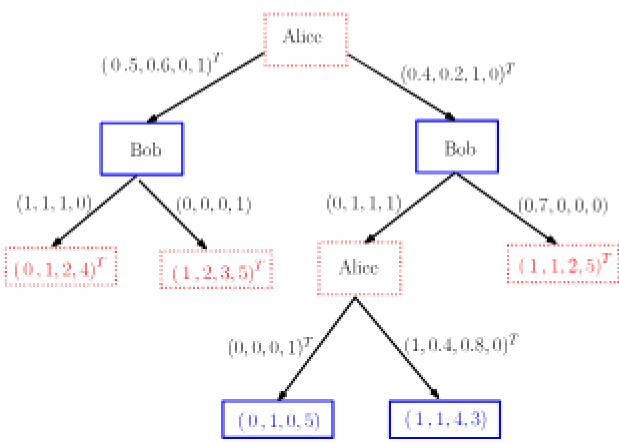
In the previous section, we had an approach to get extended formulation for $k - l$ sparsity matroids using an isomorphism with truncated transversal matroid, which in-turn was solved using network flows and a characterisation of membership of truncated transversal matroids in general. However, a similar result can be obtained as shown in [IKK+16] a more general hammer. This hammer is the relation between communication complexity and extended formulations as shown in [FFGT15]. But first, we set up basics of communication complexity, as shown in [FFGT15]

3.2.1 Deterministic Communication Protocols

Let X, Y , and Z be arbitrary finite sets with $Z \subseteq \mathbb{R}_+$, and let $f : X \times Y \to Z$ be a function. Suppose that there are two players Alice and Bob who wish to compute $f(x, y)$ for some inputs $x \in X$ and $y \in Y$. Alice knows only x and Bob knows only y. They must therefore exchange information to be able to compute $f(x, y)$. (We assume that each player possesses unlimited computational power.) The communication is carried out as a protocol that is agreed upon beforehand by Alice and Bob, on the sole basis of the function f . At each step of the protocol, one of the players has the token. Whoever has the token sends a bit to the other player, that depends only on their input and on previously exchanged bits. This is repeated until the value of f on (x, y) is known to both players. The minimum number of bits exchanged between the players in the worst case to be able to evaluate f by any protocol is called the communication complexity of f.

3.2.2 Randomized Protocols

We define a randomized protocol (with private random bits and nonnegative outputs) as a rooted binary tree with some extra information attached to its nodes. Let X and Y be finite sets, as above. Each node of the tree has a type, which is either X or Y . To each node v of type X are attached two function $p_{0,v}, p_{1,v} : X \longrightarrow [0, 1]$; to each node v of type Y are attached two functions $q_{0,v}, q_{1,v} : Y \longrightarrow [0, 1]$; and to each leaf v is attached a nonnegative vector Λ_v that is a column vector of size $|X|$ for leaves of type X and a row vector of size $|Y|$ for leaves of type Y . The functions $p_{i,v}$ and $q_{j,v}$ define transition probabilities, and we assume that $p_{0,v}(x) + p_{1,v}(x) \leq 1$ and $q_{0,v}(y) + q_{1,v}(y) \leq 1$. An example is shown in fig. 2. In the example, $|X| = |Y| = 4$.



(a)

X \ Y	y_1	y_2	y_3	y_4
x_1	0.28	0.40	1.60	1.70
x_2	0.74	0.68	0.92	1.44
x_3	1.40	0.80	3.20	2.40
x_4	4.00	4.00	4.00	5.00

(b)

Figure 2: Example of randomized communication protocol

An execution of the protocol on input $(x, y) \in X \times Y$ is a random path that starts at the root and descends to the left child of an internal node v with probability $p_{0,v}(x)$ if v is of type X and $q_{0,v}(y)$ if v is of type Y , and to the right child of v with probability $p_{1,v}(x)$ if v is of type X and $q_{1,v}(y)$ if v is of type Y . With probability $1 - p_{0,v}(x) - p_{1,v}(x)$ and $1 - q_{0,v}(y) - q_{1,v}(y)$ respectively, the execution stops at v. For an execution stopping at leaf v with vector Λ_v, the value of the execution is defined as the entry of Λ_v that corresponds to input $x \in X$ if v is of type X, and $y \in Y$ if v is of type Y . For an execution stopping at an internal node, the value is defined to be 0.

For each fixed input $(x, y) \in X \times Y$, the value of an execution on input (x, y) is a random variable. If we let $Z \subseteq \mathbb{R}_+$ as before, we say that the protocol computes a function $f : X \times Y \longrightarrow Z$ in expectation if the expectation of this random variable on each $(x, y) \in X \times Y$ is precisely $f(x, y)$. The complexity of a protocol is the height of the corresponding tree. Given an ordering $x_1, \cdots, x_m$ of the elements of X, and $y_1, \cdots, y_n$ of the elements of Y , we can visualize the function $f : X \times Y \longrightarrow Z$ as a $m \times n$ nonnegative matrix $S = S(f)$ such that $S_{i,j} = f(x_i, y_j)$ for all $(i, j) \in [m] \times [n]$. The matrix S is called the communication matrix of f . Below, as is natural, we will not always make a distinction between a function and its communication matrix.

3.2.3 Factorisation and protocols

In [FFGT15], the main theorem about non-negative rank and communication complexity is given below.

Theorem 3.18. *If there exists a randomized protocol of complexity c computing a matrix $S \in \mathbb{R}_+^{X \times Y}$ in expectation, then $lg\ rank_+(S) \leq c$. Conversely, if the nonnegative rank of matrix $S \in \mathbb{R}_+^{m \times n}$ is r, then there exists a randomized protocol computing S in expectation, whose complexity is at most $\lceil lg\ r \rceil$. In other words, if $c_{min}(S)$ denotes the minimum complexity of a randomized protocol computing S in expectation, we have*

$$c_{min}(S) = \lceil lg\ rank_+(S) \rceil$$

Since Yannakakis showed that nonnegative rank of slack matrix is equal to extension complexity, theorem 3.18 implies the following (see corollary 3.19).

Corollary 3.19. *Let P be a polytope with associated slack matrix $S = S(P)$, such that P is neither empty or a point. If there exists a randomized protocol of complexity c computing S in expectation, then $xc(P) \leq 2^c$. Conversely, if $xc(P) = r$, then there exists a randomized protocol computing S in expectation, whose complexity is at most $\lceil lg\ r \rceil$. In other words, if $c_{min}(S)$ denotes the minimum complexity of a randomized protocol computing S in expectation, we have*

$$c_{min}(S(P)) = \lceil lg\ rank_+(S) \rceil$$

We do not cover the proof of theorem 3.18 in detail (see [FFGT15]), however here is a proof sketch,

1. Given input (x, y), probability of reaching a leaf in the tree of randomized protocol is given by probability of taking “correct" branches that make the path.

2. Since protocol computes matrix S, compute $S(x, y)$ as sum of probability of getting a leaf times value at that leaf. $S(x, y) = \sum_{v \in L_x} \Lambda_v(x) P_v(x, y) + \sum_{w \in L_y} \Lambda_w(y) P_w(x, y)$, where $P_v(x, y)$ denotes probability of reaching leaf v given input (x, y).

3. P_v is a rank one matrix. This is because,

$$P_v(x, y) = \prod_{i=1}^{k} p_{\alpha_i, v_i}(x) \cdot \prod_{j=1}^{l} q_{\beta_j, w_j}(y)$$

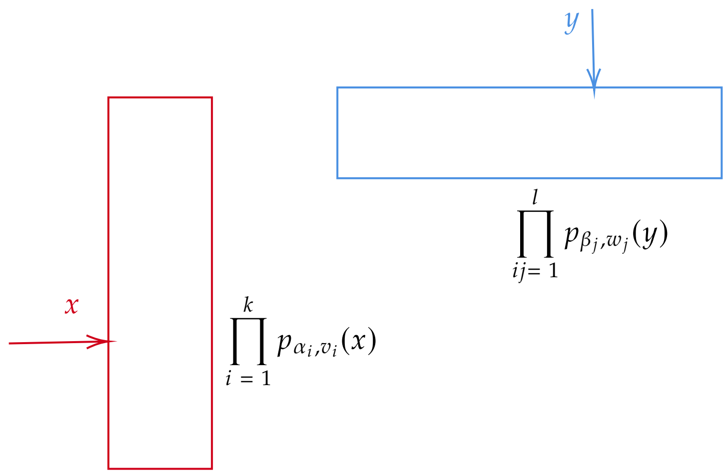


Figure 3: P_v as a rank 1 matrix

where $\alpha_i, \beta_j \in \{0, 1\}$ are fixed given v. Then, P_v is a product of column and row matrix shown in fig. 3. Hence, we get S as a sum of 2^c rank one matrices.

4. Given nonnegative rank of $S = AB$ given by r. Let $\delta_i := \sum_k A_{i,k} \leq 1$. Alice selects a column index $k \in [r]$ according to the probabilities found in row i of matrix A, sends this index to Bob, and Bob outputs the entry of B in row k and column j. With probability $1 - \delta_i$ Alice does not send any index to Bob and the computation stops with implicit output zero. Since, $\sum_k A_{i,k}B_{k,j} = S_{i,j}$, we get that this protocol computes S in expectation. Moreover, $\delta_i \leq 1$ without loss of generality, since we can scale A, B accordingly.

Now that we have this relation set up, we examine the extension complexity of $k - l$ sparsity matroids using this technique.

3.3 $k - l$ sparsity matroids (Alternate method)

The bound given by [HIH21] was predated by one with randomized communication protocol, shown by [IKK+16]. Given a polytope P, in order to obtain $xc(P)$ we need a communication protocol to compute the slack matrix of P. That is, let Alice have a facet defining inequality $a_i^T x \leq b_i$ and Bob have a vertex v_j we need a protocol to return $b_i - a_i^T v_j$.

3.3.1 Protocol when $k \geq l$

In $k - l$ sparsity matroids, inequalities are given by $x(E(X)) \leq k|E(X)| - l$ where $X \subseteq V$. Let us denote the in-degree of vertex v by $\rho(v)$. So, given a subset of vertices with Alice (facet defining inequality) and subset of edges (a vertex of polytope), the corresponding slack is given by,

$$S_{X,F} = k|X| - l - |F \cap E(X)|$$

Then, protocol in [IKK+16] is given by,

1. Alice sends a vertex $x \in X$ to Bob (Note that by vertex we mean vertex of the graph rather than that of polytope)

2. Bob orients the edges of F in such a way that $\rho(x) = k - l$ and $\rho(z) = k$ for all $z \in V \backslash \{x\}$. Then, Bob picks an oriented edge (u, v) of F uniformly at random and sends it to Alice.

3. Alice outputs $k|V| - l$ if $u \in X$ and $v \in X$, and outputs 0 otherwise.

The following key lemma will enable us to show correctness of the protocol.

Lemma 3.20. *Let k, l be integers such that $0 \leq l \leq k$, $H = (V, F)$ be a (k, l) tight graph (vertex of polytope) with $|V| \geq 2$, and $x \in V$ be an arbitrary vertex of H. Then, H has an orientation $H = (V, F)$ such that $\rho(x) = k - l$ and $\rho(z) = k$ for all $z \in V \backslash \{x\}$.*

In order to show the lemma, here is a well known result by Hakimi [Hak65],

Lemma 3.21 (Hakimi)**.** *Let $H = (V, F)$ be an undirected graph and $m(v)$ be a non-negative integer for every $v \in V$. Then, H has an orientation such that $\rho(v) = m(v)$ for all v if and only if $|F| = \sum_{v \in V} m(v)$ and $|F(X)| \leq \sum_{v \in X} m(v)$ for all $X \subseteq V$.*

Now, to prove lemma 3.20, it suffices to show condition in Hakimi's lemma. We have, $\sum_{vinV} m(v) = k(|V| - 1) + k - l = k|V| - l = |F|$, and $\sum_{v \in X} m(v) \geq k(|X| - 1) + k - l = k|X| - l \geq F(X)$. Hence, we get that the protocol is feasible. Now, to show correctness, we compute expected value of function computed by protocol.
First, we have

$$\sum_{v \in X} \rho(v) = k|X| - l$$

But, $\sum_{v \in X} \rho(v)$ counts the total in-degree of vertices in X that is, number of edges with head in X.

$$\begin{aligned} \sum_{v \in X} \rho(v) &= |\{(u, v) : v \in X, u \in X\}| + |\{(u, v) : u \notin X, v \in X\}| \\ &= |F \cap E(X)| + |\{(u, v) : u \notin X, v \in X\}| \end{aligned}$$

Hence, we get $|\{(u, v) : u \notin X, v \in X\}| = S_{X,F}$. Finally, the protocol gives expected value

$$\begin{aligned} (k|V| - l) \cdot Pr(u \notin X, v \in X) &= (k|V| - l) \cdot \frac{S_{X,F}}{|F|} \\ &= S_{X,F} \end{aligned}$$

Hence, the protocol is correct. Finally, amount of bits exchanged are,

1. $log|V|$ bits to send a vertex $x \in X$

2. $log|E|$ bits to send a random edge $e \in E$

Hence, total bits used is $log(|V||E|)$, hence extension complexity is given by $O(|V||E|)$.

3.3.2 Protocol for $k \leq l$

Similar to the previous set-up, Alice has a subset of vertices $X \subseteq V$, Bob has a set of edges.

1. Alice sends two arbitrary vertices $x, y \in X$ to Bob.
2. Bob orients the edges of F in such a way that $\rho(x) = 0, \rho(y) = 2k - l$ and $\rho(z) = k$ for all $z \in V \backslash \{x, y\}$. Then, Bob picks an oriented edge (u, v) of F uniformly at random and sends it to Alice.
3. Alice outputs $k|V| - l$ if $u \in X$ and $v \in X$, and outputs 0 otherwise.

The feasibility and correctness of this protocol also follows similarly using Hakimi's theorem and a similar analysis to the protocol of $k \geq l$. Note that here 2 vertices are sent by Alice, hence extension complexity is $O(|V|^2|E|)$.

3.4 Hitting Set of Matroid Polytopes

In the previous section, we have obtained a general hammer in the form of communication protocols to obtain extension complexity of polytopes ([FFGT15]). In this section we see a special case of this protocol where we can obtain a communication protocol in terms of hitting set of a polytope as described in [Apr22]. Hence, we get another way of getting upper bounds for extension complexity by describing communication protocols. In this section, we shall look at hitting set of polytopes, its relation with extension complexity, and getting small extended formulations for graphic and co-graphic matroids using this.

3.4.1 Hitting Set and Extension Complexity

First, let us define the hitting set of a base polytope.

Definition 3.22 (Base Polytope)**.** The base polytope of M is $B(M) = conv \chi^B : B \in \mathcal{B}$, where $\chi^B \in \mathbb{R}^E$ is the incidence vector of basis B. The following is a description of $B(M)$:

$$B(M) = \{x \in \mathbb{R}^E_+ : x(E) = rk(E), x(F) \leq rk(F) \forall \text{flat } F \subseteq E\}$$

where we write $x(F) = \sum_{e \in F} x_e$.

A base B has slack 0 with respect to flacet F iff $|B \cap F| = rk(F)$.

Definition 3.23 (Hitting family / Hitting set)**.** Let $\mathcal{V}$ be a family of bases of M. We say that V is a hitting family for M if, for any flacet F of M, there is at least one basis $B \in V$ that has full intersection with F.

Slightly abusing notation, we denote by $h(M)$ the size of the smallest hitting family for M. Notice that this does not correspond exactly to $h(B(M))$, as we are ignoring non-negativity facets, but the two numbers are the same up to an additive n factor, hence the difference is irrelevant to the

purpose of this section.
Note that independence set polytope has base polytope as a facet (adding equality constraint), hence $xc(B(M)) \leq xc(P(M))$. It also holds that $xc(P(M)) \leq xc(B(M)) + 2|E|$. This fact can be obtained by adding $0 \leq y_e \leq x_e$ constraints and taking polytope formed by y as independence polytope.

Now that we have these definitions, we state the main result of [Apr22].

Theorem 3.24. *For a matroid M of n elements and rank r,*

$$xc(B(M)) = O(h(M) \cdot n \cdot r).$$

In this report, we omit the proof of this result, but rather focus on the consequences on extension complexity, some of which are described in [Apr22].

3.4.2 Relation with extension of dual matroid

First, let us recall the definition the dual of a matroid M in terms of the base sets.

Definition (Dual Matroid)**.** The dual of a matroid M is another matroid M^* that has the same elements as M, and in which a set is independent if and only if M has a basis set disjoint from it. Alternatively, basis sets of dual matroid are complements of basis sets of M.

It is well known that given a matroid $M = (E, \mathcal{I})$,

$$r_{M^*}(S) = |S| + r_M(E \backslash S) - r_M(E)$$

In order to analyse extension complexity of dual matroid, we need to analyse the hitting set of the dual matroid. Let the set of bases of M be $\mathbb{B}$.

Lemma 3.25. *$B \in \mathbb{B}$ makes constraint corresponding to S tight in M iff $E \backslash B$ makes constraint corresponding to $E \backslash S$ tight in M^*.*

Proof. Let $x_e = 1$ iff $e \in B$, 0 otherwise. Also, since S is tight, $x(S) = r(S)$. Hence, $S \subseteq B$. Hence, we get $E \backslash B \subseteq E \backslash S$. Now, consider the indicator vector for $E \backslash S$. Let $y_e = 1$ iff $e \notin B$, 0 otherwise. So, $y(E \backslash S) = |E \backslash B|$. Hence,

$$\begin{aligned} r_{M^*}(E \backslash S) &= |E \backslash S| + r_M(S) - r_M(E) \\ &= |E \backslash S| + r_M(S) - r_M(B) \\ &= |E \backslash S| + |S| - |B| && \text{since } S \subseteq B \\ &= |E \backslash B| = y(E \backslash S) \end{aligned}$$

So, we get the desired result. The converse is obtained by considering original matroid as M^* instead of M. □

Lemma 3.26. *If $\mathcal{V}$ is a hitting set for M, $\mathcal{V}' = \{E \backslash B | B \in \mathcal{V}\}$ is a hitting set for M^*.*

Proof. For any constraint in M^*, say corresponding to set S. Some $B \in \mathcal{V}$ makes $E \backslash S$ tight. Hence, $E \backslash B$ makes S tight (using lemma 3.25), and $E \backslash B \in \mathcal{V}'$. Hence, completing the proof. □

So, given a hitting set for M, we can get an equal sized hitting set for the dual matroid, and vice-versa. So, we get

Lemma 3.27. $h(M) = h(M^*)$

3.4.3 Relation with (co)-graphic matroid polytope

In this section, we see that as shown in [Apr22], graphic matroid polytopes admit a small hitting set and the upper bound obtained on extension complexity matches the best known bounds. As a corollary using lemma 3.27, we get a small extended formulation for the co-graphic matroid as well. For the graphic matroid base polytope (denote as $P(G)$), we have constraints given by

$$x(E(U)) \leq |U| - 1 \qquad \phi \neq U \subset V$$
$$x(E(V)) = |V| - 1$$

where $E(U)$ is set of edges with both endpoints in U. Let the graph $G = K_n$. Now, for any other G', we get that base polytope for G' is a facet of G, hence adding $x_e = 0$ constraints enables the extension complexity bound to carry over. So, considering $G = K_n$ is enough. Let the edge set of star graph with edges (v, u) for all $u \in V - \{v\}$ be denoted as δ_v. Now, we have the following claim,

Lemma 3.28. *$\mathcal{V} = \{\delta_v | v \in V\}$ is a hitting set for $P(G)$.*

Proof. For any $U \subseteq E$, with $u \in U$ the facet inducing inequality is given by,

$$x(E(U)) \leq |U| - 1$$

Putting the vector x corresponding to δ_u, we get $x(E(U)) = \sum_{e \in E(U)} e \in \delta(u) = d_u = |U| - 1$. Moreover, all star graphs are indeed spanning trees, i.e. base sets, hence showing the result. □

So, using lemmas 3.28 and 3.24, we get that for a graph $G = (V, E)$ with $|V| = n$, $h(M) = n$, number of elements = $\binom{n}{2}$, rank $r = n - 1$. Hence, we get an extended formulation of size $O(n^4)$. However, this approach can be made more efficient as shown in [Apr22] by bypassing a step in the derivation of theorem 3.24 to get an $O(n^3)$ extension. However, we omit this modification in this report.

Another approach using hitting sets examined in [Apr22] is extension for regular matroids. However we come back to this in the next section after setting up some machinery for handling 2-sums of matroids.

3.5 2-sums of Matroid Polytopes

Before diving into why this operation matters, let us define direct sum, 2-sum of matroids.

Definition 3.29 (Direct Sum)**.** Consider matroids $M_1 = (E_1, \mathbb{B}_1)$ and $M_2 = (E_2, \mathbb{B}_2)$, with non-empty base sets. If $E_1 \cap E_2 = \phi$, we can define the direct sum $M_1 \oplus M_2$ as the matroid with ground set $E_1 \cup E_2$ and base set $B_1 \times B_2$.

Definition 3.30 (2-sums)**.** Consider matroids $M_1 = (E_1, \mathbb{B}_1)$ and $M_2 = (E_2, \mathbb{B}_2)$, with non-empty base sets. If $E_1 \cap E_2 = \{p\}$, we can define the 2-sum $M_1 \oplus_2 M_2$ as the matroid with ground set $E_1 \cup E_2 - p$ and base set $\{B_1 \cup B_2 - p : B_1 \in \mathbb{B}_1, B_2 \in \mathbb{B}_2, p \in B_1 \Delta B_2\}$

where Δ denotes symmetric difference. In [ACF18] the following lemma is shown,

Lemma 3.31. *Let $M_1(E_1, \mathbb{B}_1), M_2(E_2, \mathbb{B}_2)$ be matroids with $E_1 \cap E_2 = \{p\}$ and let $M = M_1 \oplus_2 M_2$. Then $B(M)$ is linearly isomorphic to $B(M_1) \times B(M_2) \cap \{x \in \mathbb{R}^{E1 \uplus E2} : x_{p_1} + x_{p_2} = 1\}$, where $E_1 \uplus E_2 = E_1 \cup E_2 \cup \{p_1, p_2\} - p$ is the disjoint union of E_1 and E_2, with p_1 and p_2 corresponding to $p \in E_1$ and $p \in E_2$ respectively.*

3.5.1 Connection with hitting set

In this section we show the following lemma,

Lemma 3.32. *Consider matroids $M_1 = (E_1, \mathbb{B}_1), M_2 = (E_2, \mathbb{B}_2)$ with $E_1 \cap E_2 = \{p\}$, then, $h(M_1 \oplus_2 M_2) \leq h(M_1) + h(M_2)$.*

Proof. Consider the smallest hitting sets for $P(M_1)$ and $P(M_2)$ (say $\mathcal{V}_1$, $\mathcal{V}_2$). Then, consider the polytope P given by $B(M_1) \times B(M_2) \cap \{x \in \mathbb{R}^{E_1 \uplus E_2} : x_{p_1} + x_{p_2} = 1\}$. By lemma 3.31, P and $P(M_1 \oplus_2 M_2)$ are affinely isomorphic. WLOG, $M_1 \oplus_2 M_2$ is non-empty, hence there exist bases $B_1 \in \mathbb{B}_1$, $B_2 \in \mathbb{B}_2$ such that $p \in B_1$, $p \notin B_2$. Let there exist bases $p \notin B_1' \in \mathbb{B}_1$, $p \in B_2 \in \mathbb{B}_2$. Then, consider the set $\{[V_1, B_2'] : p \in V_1 \in \mathcal{V}_1\} \cup \{[V_1', B_2] : p \notin V_1' \in \mathcal{V}_1\} \cup \{[B_1, V_2] : p \notin V_2 \in \mathcal{V}_2\} \cup \{[B_1', V_2'] : p \in V_2' \in \mathcal{V}_2\}$. Then for any facet defining constraint in $B(M_1)$, we get some base $B_1 \in \mathcal{V}_1$ such that B_1 is tight with respect to the constraint. Then, an appropriate base in $\mathcal{V}_2$ can be appended to B_1 to make the constraint tight in the new polytope. Hence, we get a hitting set of size $h(M_1) + h(M_2)$.
If all bases in M_1 contain p (or do not contain p), then there exists a base in M_2 not containing (containing) p. A similar idea still works since for all constraints that can become tight by a base set in M_2 containing p and not by base sets not containing p, the constraint will not be facet defining in the new polytope. □

3.5.2 Connection with regular matroids

First, recall that regular matroids are defined as linear matroids, when the matrix is totally unimodular. It is well known that $O(n^6)$ extension complexity holds for regular matroids as shown in [AF19]. This result relies on using seymour's decomposition,

Theorem 3.33 (Seymour's decomposition)**.** *A matroid is regular if and only if it is obtained by means of* $1-, 2-$ *and 3-sums, starting from graphic and cographic matroids and copies of a certain 10-elements matroid* R_{10}.

Hence, an approach to getting small extension complexity for regular matroids is showing that, $xc(P(M_1 \oplus_t M_2)) \leq xc(P(M_1)) + xc(P(M_2))$ for $t = 1, 2, 3$. However, for $t = 1$ it holds trivially since its a cartesian product. Similarly for $t = 2$, its the intersection of cartesian product with a single hyperplane. 3-sum is handled by the rest of the paper.
But, we can have two other approaches as shown in [Apr22],

1. Show that $h(M_1 \oplus_t M_2)$ is small for $t = 1, 2, 3$. For $t = 1, 2$ we know that $h(M_1 \oplus_t M_2) \leq h(M_1) + h(M_2)$ as shown in lemma 3.32. But, for 3-sum it is not clear how to approach this.

2. Analyse size of hitting sets for regular matroid bypassing seymour's decomposition. Similar to proof shown for graphic matroids, we only need to consider the "maximal" regular matroids. So, we only need to consider complete regular matroids which is known to have atmost $r(r+1)/2$ elements, if rank is r.

3.6 Extension of Polygons

In this section we examine the extension complexity of polygons as shown in [FRT12]. This paper shows 3 main results of which, we will discuss the first two.

1. Extension complexity of regular n-gons is $O(\log n)$.

2. Extension complexity of generic n-gons is lower bounded by $\sqrt{2n}$.

3. There exist n-gons whose vertices lie on an $O(n) \times O(n^2)$ integer grid with extension complexity $\Omega(\sqrt{\frac{n}{\log n}})$

3.6.1 $O(\log n)$ extension for regular polygons

In this section we show an $O(\log n)$ extension for regular n-gons. The construction of the extended formulation is very similar to binary search.

Theorem 3.34. *Let P be a regular n-gon in $\mathbb{R}^2$. Then $xc(P) = O(\log n)$.*

We describe the sketch of the proof here (see [FRT12] for details).

Without loss of generality, we may assume that the origin is the barycenter of P. After numbering the vertices of P counter-clockwise as $v_1, \cdots, v_n$, we define a sequence $l_0, \cdots, l_{q-1}$ of axes of symmetry of P, as

1. Initialize i to 0, and k to n. While $k > 1$, repeat the following steps:

    (a) define l_i as the line through the origin and the midpoint of vertices $v_{\lceil \frac{k}{2} \rceil}$ and $v_{\lceil \frac{k+1}{2} \rceil}$

    (b) replace k by $\lfloor \frac{k+1}{2} \rfloor$

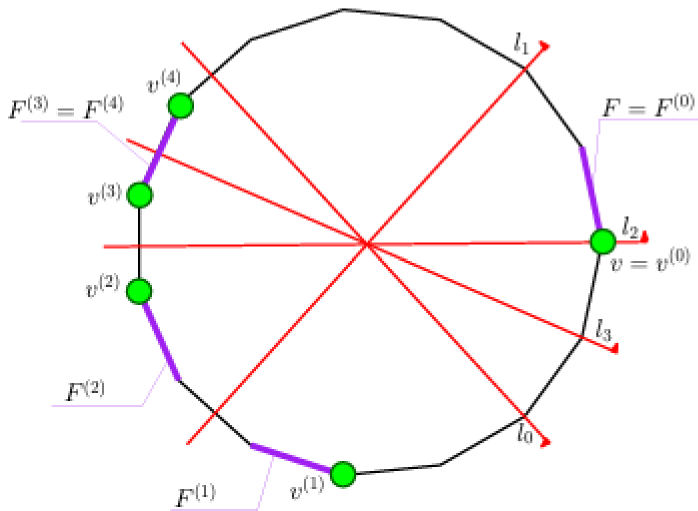


Figure 4: Construction of folding sequence given an n-gon

(c) increment i

For each $i = 0, \cdots, q-1$, one of the two closed halfplanes bounded by l_i contains v_1. We denote it l_i^+ . We denote the other by l_i^- ($q = O(\log n)$).

Now, consider a vertex v of P. We define the folding sequence $v^{(0)}, v^{(1)}, \cdots, v^{(q)}$ of v as follows. We let $v^{(0)} := v$, and for $i = 0, \cdots, q-1$, we let $v^{(i+1)}$ denote the image of $v^{(i)}$ by the reflection with respect to l_i if $v^{(i)}$ is not in the halfspace $l + i$, otherwise we let $v^{(i+1)} := v^{(i)}$. Similarly define the reflection of a facet F given by $a^T x \leq b$ as conditional reflection of $a^{(i)}$ along l_i. Initially, start with any facet say $a^T x \leq \beta$ (facet i is $(a^{(i)})^T x \leq \beta$). Since final vertex of any folding sequence is v_1, get that final facet corresponds to edge $[v_1, v_2]$ or $[v_n, v_1]$. Upon analysis of slack, we get,

$$\beta - (a^{(i)})^T v^{(i)} = \beta - (a^{(i+1)})^T v^{(i+1)} \qquad \text{if } v^{(i)}, a^{(i)} \text{ lie in same side of } l_i$$
$$\beta - (a^{(i)})^T v^{(i)} = \beta - (a^{(i+1)})^T v^{(i+1)} + 2d(a^{(i)}, l_i)d(v^{(i)}, l_i) \qquad \text{otherwise}$$

Now, we construct factorisation of slack as,

$$S(P) = TU$$

In the left factor, row corresponding to facet F is of form, $(t_0, \cdots, t_{q-1})$, where $t_i := (\sqrt{2}d(a^{(i)}, l_i), 0)$ if $a^{(i)}$ is not in l_i^+ and $t_i := (0, \sqrt{2}d(a^{(i)}, l_i))$ otherwise. Similarly define column corresponding to corner v as $(u_0, \cdots, u_{q-1})^T$, where $u_i := (0, \sqrt{2}d(v^{(i)}, l_i))^T$ if $v^{(i)}$ is not in l_i^+ and $u_i := (\sqrt{2}d(v^{(i)}, l_i), 0)^T$ otherwise. Now, slack of $v^{(q)}$ with respect to $F^{(q)}$ is always 0 (since it is the $\log n$ step of this process). Hence, applying telescopic sum on both sides of the slack equations, we get the desired result.

3.6.2 Lower bound of $\sqrt{2n}$ for generic n-gons

This bound uses algebraic methods for bounding extension complexity.

Definition 3.35. A subset X of L is said to be algebraically independent over K if no non-trivial polynomial relation with coefficients in K holds among the elements of X. The transcendence degree of the field extension L/K is defined as the largest cardinality of an algebraically independent subset of L over K. It is also the minimum cardinality of a subset Y of L such that $L/K(Y)$ is algebraic.

Theorem 3.36. *If P is a generic convex n-gon in $\mathbb{R}^2$ then $xc(P) \geq \sqrt{2n}$*

Recall that a generic convex n-gon is defined as the coordinates of its vertices being algebraically independent over $\mathbb{Q}$. Algebraically independent means there is no non-zero polynomial over $\mathbb{Q}$ with variables $x_1, x_2, \cdots, x_{2n}$ such that putting in coordinates of vertices makes it 0.
Let $\alpha_1, \cdots, \alpha_{2n}$ denote n vertices of P, listed in any order. So, $X = \{\alpha_1, \cdots, \alpha_{2n}\}$ is alg. independent over $\mathbb{Q}$.

Let P be the projection of a d-dimensional polytope Q with k facets. Without loss of generality, assume that P is projection of Q along first two coordinates (if not, a change of basis gives a new polytope Q' whose first 2-coordinates project to P).

Consider a linear description of Q. We need k inequalities for facets and each inequality has $d+1$ coefficients. Let us denote the coefficients as $\beta_{i,j}$ for $i \in \{1, \cdots, k\}, j \in \{1, \cdots, d+1\}$. Then by cramer's rule, the corners have coordinates as rational function of $\beta_{i,j}$ coordinates. Hence, α_j are rational function of $\beta_{i,j}$. Hence each α_j belongs to the extension field $L(\mathbb{Q}, \beta)$. Since X is algebraically independent over $\mathbb{Q}$, $X \subseteq L$, atmost transcedence degree of $L/\mathbb{Q}$ is atleast $2n$. But, atmost $k(d+1)$ algebraically independent elements have been added to field $\mathbb{Q}$. Hence transcedence degree is upper bounded by $k(d+1)$. (If not, there exists an element of $\mathbb{Q}$ in the chosen set, implying that setting all other coefficients to 0 in the polynomial gives us a non-trivial polynomial attaining 0 on the chosen points). Hence, $k(d+1) \geq 2n$. But, since polytopes are bounded, we get $d+1 \leq k$. Hence, $k \geq \sqrt{2n}$.

3.7 Exact Spanning Tree, Matchings

In this section, we discuss the progress in exact matchings made by [JSY]. However, the primary objective of this inquiry is to ask the following questions.

1. What is the extension complexity of the exact spanning tree polytope i.e. polytope of spanning trees having exactly k-red edges (Alternatively the polytope of spanning trees having odd red edges)?

2. What is the extension complexity of polytope of spanning trees having weight exactly W?

Both of the above questions remain open, however we discuss some small observations here.

3.7.1 Exact weighted spanning tree

The first observation to be noted is that the problem is NP-complete. There is a simple reduction from subset-sum. Let there be a set of objects with weights $\{w_1, \cdots, w_n\}$ and target weight W. Then, consider a graph given by fig. 5, and target weight $T = W + (n+1)\varepsilon$.

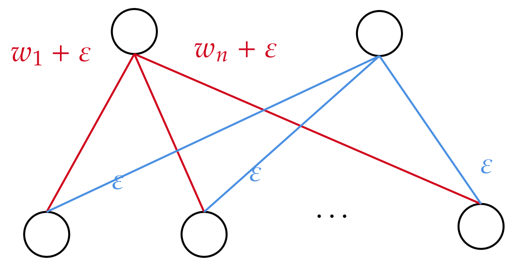


Figure 5: Reduction from subset sum

Lemma 3.37. *The graph shown in fig. 5 has a spanning tree of weight $W + (n+1)\varepsilon$ iff there is a subset of $\{w_1, \cdots, w_n\}$ with target sum W.*

Proof. Let there exist a subset satisfying target sum. Then choose red edges for all elements in that subset, and choose blue edges for all other elements. Choose an additional blue edge for exactly one element of chosen subset to complete the spanning tree. It is clear that the weight of the tree is $W + (n+1)\varepsilon$.
Let there exist a spanning tree with target weight. Since the spanning tree must have $n+1$ edges, total degree must be $2n+2$. Now, every edge chosen increases the degree of upper vertices in the figure by 1, hence degree of upper vertices is $n+1$. Now, every vertex in bottom row must have degree atleast one and combined degree of n vertices is $n+1$. Hence $n-1$ vertices have degree 1, and exactly one vertex has degree 2 in the bottom row. Let there be k red edges incident on the set S in bottom row and $n-k+1$ blue edges. So, $\sum_{i \in S} w_i + (n+1)\varepsilon = W + (n+1)\varepsilon$, hence $W = \sum_{i \in S} w_i$. □

Hence, for general weights we cannot expect a small extended formulation since that would give us an LP based algorithm for subset sum running in PTIME. Another contributing factor to why such a result cannot be expected is that the $0-1$ knapsack polytope has extension complexity atleast $2^{\sqrt{n}}$ as shown in [PV13] (knapsack polytope is defined as convex hull of points in $\{0,1\}^n$ such that $a^T x \leq b$). However, we can expect a small extended formulation for small weights.
The obvious first attempt is adding a target weight constraint to the spanning tree polytope. That is,

$$\begin{aligned} x(E(U)) &\leq |U| - 1 & \phi \neq U \subset V \\ x(E(V)) &= |V| - 1 \\ \sum_e x_e w_e &= W \end{aligned}$$

However this attempt fails because adding the constraint $\sum_e x_e w_e = W$ makes the polytope non-integral. Consider the following example (see fig. 6) where values of w_e are indicated in black and those in x_e are red, with target weight $W = 9$

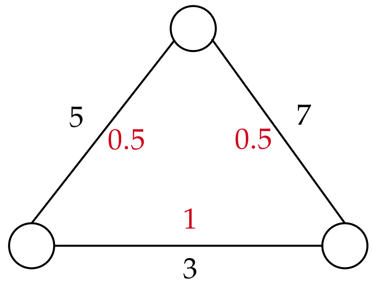


Figure 6: Counterexample for obvious LP formulation

We can see that no spanning tree of weight 9 exists, however corresponding LP gives a feasible non-integral solution.

3.7.2 Exact or odd spanning tree

Since the case with weights is not approachable, an alternative to be considered is the case with no weights on edges and choosing spanning trees with exactly k red edges, in a graph colored with red or blue edges. We know (due to Balas) that for polytopes P_i, $i \in \{1, \cdots, n\}$,

Theorem 3.38. $xc(\bigcup_i P_i) \leq \sum_i xc(P_i)$

This result is simple to obtain by taking points x_i in polytopes P_i and taking their convex combination to get new point x. Now, consider the polytope of odd spanning trees. Let P_i denote polytope of spanning trees with exactly i red edges.

$$xc(P_{odd}) \leq \sum_{\text{odd } i \in \mathbb{N}} P_i$$

A related question was tackled by Svennson et. al. [JSY], where it was shown that extension complexity of odd perfect matchings in bipartite graphs is exponential, hence the same result holds for exact matchings as well. This paper has a 2-step approach to solving this problem,

1. Giving an inequality description for odd perfect matchings (albeit with exponentially many constraints).
2. Relax the polytope and lower bound its extension complexity.

The approach used for lower bounding the extension complexity is similar to the one used by Rothvoss [Rot17] in lower bounding extension complexity of the perfect matching polytope.
Back to tackling this question, the problem of finding exactly k red edges, odd red edges has a polynomial time algorithm. So, it is natural to expect a small extended formulation for this problem. However, this remains open.

4 Discussion

In conclusion the set of open problems relevant to this survey are,

1. Extension complexity of dilworth truncation in general matroids. As a first step, consider modified constraints as $x(S) \leq 2r(S) - 1$. For graphic matroids, we see that $2|S| - 3$ is a $k - l$ sparsity matroid, hence it has a small extension complexity (One possible approach can be using the hitting set).

2. Extension complexity of odd red spanning free polytope. First, we need to find an LP formulation for this followed in order to get started.

3. Alternative method for extension of regular matroid using hitting set.

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