Extension Complexity for Matroid Union*

Ayush Agarwal
ayushagarwal@cse.iitb.ac.in
IIT Bombay

Soham Joshi
sohamjoshi@cse.iitb.ac.in
IIT Bombay

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Abstract

Given a matroid, or any polytope, its extension complexity was characterised Yannakakis [Yan88]. The extension complexity of matroid intersection is trivially polynomial given that the orignal matroids have polynomial extension complexity. This is because the constraints describing matroid intersection have a nice formulation. However, the same problem for union of matroids remains open. In this work, we show that in fact, the union of matroids also a polynomial upper bound to the extension.

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Contents

1 Introduction

The extension complexity of a polytope P is the smallest number of facets among convex polytopes Q that can have P as a projection. In this case, Q is called an extended formulation of P and it may have a higher dimension than P. The extension complexity of a polytope is a natural question to ask since many problems can be encoded as a combination of linear constraints. If this admits a small extended formulation, then standard techniques like linear program solvers can be used to solve the problem. In 1980s, Swart [Swa87] attempted to prove that there is a polytope with only polynomial many facets that projects down to the TSP polytope. Note that a polynomial size extended formulation for the TSP polytope would imply $P = NP$. The purported linear programs were extremely complicated to analyze. However, in a breakthrough paper, Yannakakis [Yan88] refuted all such attempts by showing that every symmetric linear program (LP) for the TSP polytope has exponential size. Since the proposed LP by Swart was symmetric, that refuted Swart's proof.

However, the question of extension complexity of non-symmetric LPs remained open. However, in 2010, Kaibel, Pashkovich, and Theis [KPT10] gave examples of polytopes that do not have polynomial size symmetric extended formulations, but that do have polynomial size asymmetric extended formulations. This rekindled interest in the lingering question. In another breakthrough paper in 2012, Fiorini, Masser, Pokutta, Tiwary, and de Wolf [FMP+12] finally proved that the TSP polytope does not admit any polynomial size extended formulation, symmetric or not.

Further investigation was done into the extension complexity of various other polytopes, for instance several kinds of matroid polytopes. In particular, we know that perfect matchings for a general graph can be found in polynomial time via edmond's algorithm. Hence, it was quite a surprise when Rothvoss [Rot17] showed that the perfect matching polytope (convex hull of all perfect matchings of the complete graph K_n) does not admit any polynomial size extended formulation.

In the case of matroids, most of the well-known types like graphic matroid, transversal matroid, partition matroid, etc. have a small extension complexity. Recently it was shown by Aprille & Fiorini [AF19] that independence set polytopes of regular matroids have small extension complexity. Moreover, Iwata, Kamiyama, Katoh, Kijima, and Okamoto [IKK+14] have shown that $k - l$ sparsity matroids have polynomial extension. It is known that union of matroids is also a matroid, and it can be found efficiently (see [S+03]). So, if the initial matroids have a small LP description, it is natural to expect that their union also has a small description. In this work, we show that this intuition is indeed correct, that is, given matroids with small extension, their union also admits a small extended formulation.

2 Preliminaries

Matroid theory originated in the middle of the 1930s. There is a huge literature on matroids by now. For an introduction, see for example the excellent textbooks of Oxley [Oxl06] or Schrijver [S+03]. Moreover, our problem also deals with the extension complexity of matroids, for which refer the excellent lecture notes by Li [Li18] as an introduction. Below we give some basic definitions and facts about matroids and extension complexity of polytopes.

2.1 Matroid

A matroid M is a pair $M = (E, I)$, where E is the finite ground set and $I \subseteq P(E)$ is a nonempty family of subsets of E that satisfies the following two axioms.

1. Closure under subsets. For every $I \in \mathcal{I}$ and $J \subseteq I$ we have $J \in \mathcal{I}$.
2. Augmentation property. For every $I, J \in \mathcal{I}$ where $|I| < |J|$, there is an $j \in J$ such that $I \cup j \in \mathcal{I}$.

We denote $m = |E|$ throughout the paper. The sets in I are called the independent sets of M. An inclusion-wise maximal set $B \in \mathcal{I}$ is called a base. Note that by the augmentation property, all base sets have the same size. Let $\mathcal{B} \in \mathcal{I}$ denote the collection of base sets.

2.1.1 Matroid Rank

. Motivated by Linear Algebra, there is a rank-function of a matroid that is defined for every subset $A \subseteq E$ as the size of the largest independent set that is contained in A,

$$rank(A) = max\{|I| \mid I \in \mathcal{I} \text{ and } I \subseteq A\}$$

The size of every maximal independent set is rank(E). This number is called the rank of M. The matroid problem is to compute a maximal independent set.

An important property of the rank-function is its submodularity. In general, a function $f : P(E) \to \mathbb{R}$ is called submodular, if for any sets $S, T \subseteq E$, we have

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$$

.

2.1.2 Matroid Polytope

The polytopes we consider in this paper are convex polytopes defined as the convex hull of finitely many points in $\mathbb{R}^m$. Any convex polytope P can be described as the intersection of halfspaces, i.e., as $P = \{x \in \mathbb{R}^m | Ax \leq b\}$, for some matrix $A \in \mathbb{R}^{k \times m}$ and vector $b \in \mathbb{R}^k$. A face of the polytope P is the set of points in P minimizing or maximizing a linear function. If the polytope P is described by $Ax \leq b$, then any face of P can be described as $\{x \in P \mid A'x \leq b'\}$, where $(A' \; b')$ is some subset of the rows of $(A \; b)$. With every matroid, there is an associated matroid polytope. This polytope is

crucial for our arguments.
For a set $I \in E$, its characteristic vector $x^I \in R^E$ is defined as

$$x = \begin{cases} 1, & \text{if } e \in I \\ 0, & \text{otherwise} \end{cases}$$

For any collection of sets $A \subseteq P(E)$, the polytope $P(A) \subset R^E$ is defined as the convex hull of the characteristic vectors of the sets in A,

$$P(A) = conv\{x^I \mid I \in A\}$$

For a matroid $M = (E, I)$, its matroid polytope is defined as $P(I) \subset \mathbb{R}^E$, i.e., the convex hull of the characteristic vectors of the independent sets. The points $\{x^I \mid I \in \mathcal{I}\}$ are the corners of the matroid polytope $P(I)$. Edmonds [Edm70] gave a simple description of this polytope which uses the rank function of the matroid (see also [Sch03]). For convenience, we define for any $x \in \mathbb{R}^E$ and $S \subseteq E$,

$$x(S) = \sum_{e \in S} x_e$$

Lemma 2.1 ([Edm03])**.** *For a matroid (E, I) with rank function r, a point $x \in \mathbb{R}^E$ is in $P(I)$ iff*

$$x_e \geq 0 \; \forall \; e \in E(1) \tag{1}$$

$$x(S) \leq r(S) \; \forall \; S \subseteq E \tag{2}$$

It is easy to see that any 0-1 corner of the polytope given by (1) and (2) corresponds to an independent set in $\mathcal{I}$. The nontrivial part is to show that the described polytope does not have a non-integral corner. Let $\mathcal{B}$ be the family of base sets of the matroid $(E, \mathcal{I})$. Let n be the rank of the matroid, i.e., the size of any base set. The matroid base polytope, defined as $P(B)$, is clearly a face of the matroid polytope $P(I)$. Putting the following equation together with (1) and (2) will give a description of $P(B)$,

$$x(E) = n \tag{3}$$

2.1.3 Matroid Intersection

The matroid intersection problem is, given two matroids $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$ over the same ground set, compute a maximum size set in $I_1 \cap I_2$, the common independent sets. Note that in general $(E, I_1 \cap I_2)$ is not a matroid anymore.

2.1.4 Matroid Intersection Polytope

The intersection of two matroids also has an easy polytope description: Edmonds [Edm70] showed a surprising result that one can describe the matroid intersection polytope $P(I_1 \cap I_2)$ just by putting together the constraints of the two matroid polytopes $P(I_1)$ and $P(I_2)$ (see also [S$^+$03])

Theorem 2.2 ([Edm03]). *For two matroids (E, I_1) and (E, I_2),*

$$\mathcal{P}(M_1 \cap M_2) = \mathcal{P}(M_1) \cap \mathcal{P}(M_2)$$

That is, a point $x \in \mathbb{R}^E$ is in the polytope $\mathcal{P}(M_1 \cap M_2)$ iff

$$x_e \geq 0 \ \forall \ e \in E \tag{4}$$

$$x(S) \leq r_1(S) \ \forall \ S \subseteq E \tag{5}$$

$$x(S) \leq r_2(S) \ \forall \ S \subseteq E, \tag{6}$$

where r_1 and r_2 are the rank functions of the two matroids, respectively.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the families of base sets of the matroids $(E, I1)$ and $(E, I2)$, respectively. Note that there can be a common base set only if the two matroids have same rank, say n. To obtain the common base polytope $P(B_1 \cap B_2)$ one just needs to put the constraint 3 together with inequalities 4, 5 and 6.

2.1.5 Chaining tight sets

In this section, we state a lemma (see [S+03]) which expresses any tight constraint as a linear combination of tight constraints obtained by a chain of sets.

Definition 2.3 (Tight sets). Let $\mathcal{M}$ be a matroid with the rank function r. Let $\boldsymbol{x} \in \mathcal{P}(\mathcal{M})$ (matroid polytope). We call a set S a tight set of $\boldsymbol{x}$ with respect to r if $x(S) = r(S)$

Lemma 2.4 (Uncrossing operation). *Let $\mathcal{M}$ be a matroid with rank function r, and let $\boldsymbol{x} \in \mathcal{P}(\mathcal{M})$. If S and T are tight sets of $\boldsymbol{x}$ with respect to r, then so are $S \cup T$ and $S \cap T$.*

Lemma 2.5 (Maximal Chain of Tight Sets). *Let $\mathcal{M}$ be a matroid with rank function r, and let $\boldsymbol{x} \in \mathcal{P}(\mathcal{M})$. Let $\mathcal{C} = \{C_1, \cdots, C_k\}$ with $\phi \subset C_1 \subset \cdots \subset C_k$ be an inclusion-wise maximal chain of tight sets of $\boldsymbol{x}$ with respect to r. Then every tight set T of $\boldsymbol{x}$ with respect to r must satisfy $\chi_T \in span\{\chi_C : C \in \mathcal{C}\}$*

2.1.6 Partition Matroid

Partition matroid is a matroid in which E is partitioned into (disjoint) sets $E_1, E_2, .., E_l$ and

$$I = \{X \subseteq E : |X \cap E_i| \ \leq k_i \ \forall \ i = 1, .., l\},$$

for some given parameters $k_1, .., k_l$

2.1.7 Partition Matroid Polytope

For a given partition matroid $P = (E, \mathcal{I})$ and partition $E_1, E_2, .., E_l$, a point $x \in R^E$ is in the polytope $P(I_1 \cap I_2)$ iff

$$x_e \geq 0 \ \forall \ e \in E \tag{7}$$

$$x(E_i) \leq r_1(E_i) \ \forall \ i \tag{8}$$

2.1.8 Matroid Union

Matroid union of $M_1, M_2, M_3, .., M_k$ is defined as

$$M = M_1 \vee M_2 \vee .. \vee M_k = (\cup_{i=1}^{k} S_i, \mathcal{I} = \{\cup_{i=1}^{k} I_i | I_i \in \mathcal{I}\})$$

We can show that M is a matroid, and derive the rank function for it which is given by,

$$r_m(U) = \min_{T \subseteq U} \left[U \backslash T + \sum_{i=1}^{k} r_{M_i}(T \cap S_i) \right]$$

2.2 Extension Complexity

Given a polytope $P \subseteq \mathbb{R}^n$ as a convex hull of finitely many points (called vertices) in $\mathbb{R}^n$, by Minkowski- Weyl theorem, it is equivalent to a bounded polyhedron Q that is described by a system of linear equalities. Let $|Q|$ denote the number of linear inequalities in Q's description. There might be different LP formulations for the same polytope. Now, LP solvers can give a solution to an optimization problem on this polytope if the LP formulation is polynomial in $|Q|$. Hence, there are two directions to investigate, as given in [Li18].

1. Find another description Q' with polynomially many inequalities;
2. Find a higher dimension polytope (bounded polyhedron) $H \subseteq \mathbb{R}^k$ (where $k > n$) such that it projects to P, and H has a $poly(n)$ description.

In some cases, it can be shown that 1 is not possible by showing that any description in $\mathbb{R}^n$ has exponential size, hence we can only turn to 2. Note that any optmization problem over P can be done by optimizing the same objective function over H.

Definition 2.6. Let $P \subseteq \mathbb{R}^n$ be a polytope, a polytope $H \subseteq \mathbb{R}^k$ in a higher dimensional space is called an extended formulation of P if $\pi(H) = P$, where $\pi : \mathbb{R}^k \to \mathbb{R}^n$ is the projection map. The extension complexity of P denoted by $xc(P)$, is defined to be :

$$xc(P) := \min_{Q:\ Q \text{ is an extended formulation of } P} |Q|$$

We say P has a compact formulation if $xc(P) = poly(n)$.

Now, linear programs with compact formulations as defined above can be solved in `PTIME` by using an LP solver on the extended formulation of the LP. Note that if a polytope does not have a compact formulation, that does not imply that the problem cannot be solved in `PTIME`, one of the most famous examples being the matching polytope. Finding a maximum matching for a general graph has a polynomial time algorithm, but it has been shown recently that the matching polytope has no compact formulation [Rot17].

2.2.1 Non-negative rank characterization

A characterization of the extension complexity was given by [Yan88] in terms of the non-negative rank of the "slack matrix" associated with the polytope.

Definition 2.7 (Slack matrix)**.** Given a polytope $P \subseteq \mathbb{R}^n$ suppose it is described by $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ where $A_{m\times n}$ is a matrix and $b \in \mathbb{R}^m$ is a vector. Let the set of vertices of P be $\{z_1, z_2, \cdots, z_k\}$. The slack matrix of P under the above description is defined to be the matrix

$$S_{m\times k} := (b - Az_1, b - Az_2, \cdots, b - Az_k) \in \mathbb{R}^{m\times k}_{\geq 0}$$

Definition 2.8 (Non-negative rank)**.** Given a matrix $S_{m\times k}$, its non-negative rank, denoted by $\text{rank}_+(S)$, is defined to be the least integer $r \in \mathbb{N}$, such that $S = L_{m\times r}R_{r\times k}$ where L and R are two non-negative matrices.

Now, with these definitions, we state Yannakakis's characterization of the extension complexity of a polytope.

Theorem 2.9 ([Yan88])**.** *For any polytope $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ whose dimension ≥ 1, let S be its slack matrix, one has*

$$xc(P) = rank_+(S)$$

3 Extension for union of two matroids

In this section, we show how to obtain a small extended formulation (extension) for the matroid union polytope, given small extensions for two matroid polytopes. To begin, we shall prove that this follows for the matroid intersection polytope trivially.

3.1 Matroid Intersection has small extension

Suppose you have 2 matroids $M_1 = (S_1, \mathcal{I}_1)$ and $M_2 = (S_2, \mathcal{I}_2)$, for which we have a compact formulation. Now, we wanted to give the extension complexity for $M_1 \vee M_2$. Edmond showed that $\mathcal{P}(M_1 \wedge M_2) = \mathcal{P}(M_1) \cap \mathcal{P}(M_2)$ where $\mathcal{P}$ represents the corresponding polytope. Hence, the matroid intersection polytope is given by taking constraints $\mathcal{P}(M_1)$ and $\mathcal{P}(M_2)$ together. Let $\boldsymbol{x(S)}$ denote the projection of $\boldsymbol{x}$ onto the coordinates of S, where S is a set of variables.

Lemma 3.1 (Extension of Matroid Intersection)**.** *Given matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ which have a compact formulation, their matroid intersection polytope has a compact formulation*

Proof. Let $M_1 = (S, \mathcal{I}_1)$, $M_2 = (S, \mathcal{I}_2)$ correspond to polytopes $\mathcal{P}(M_1)$ and $\mathcal{P}(M_2)$, where $P(M_1)$, $P(M_2)$ both have constraints over the variables $X = \{x_i : i \in S\}$. Since, both of them have a compact formulation, let $Q(M_1)$, $Q(M_2)$ denote their extended formulations, where $|Q(M_1)| = poly(|S|)$ and $|Q(M_2)| = poly(|S|)$. Then consider another polytope $Q(M_1 \wedge M_2)$ given by taking constraints of $Q(M_1)$, $Q(M_2)$. Now, $Q(M_1 \wedge M_2)$ clearly has $poly(|S|)$ constraints. Now, we show that $Q(M_1 \wedge M_2)$

is an extension of $\mathcal{P}(M_1 \wedge M_2)$. Let set of variables appearing in $Q(M_1), Q(M_2)$ be denoted as X_1, X_2

First, we show if $x \in Q(M_1 \wedge M_2)$ then $x(X) \in \mathcal{P}(M_1 \wedge M_2)$. $x \in Q(M_1 \wedge M_2)$ implies $x(X_1) \in Q(M_1)$ and $x(X_2) \in Q(M_2)$. Since $X \subseteq X_1, X_2$, hence, $x(X_1)(X) = x(X) \in \mathcal{P}(M_2)$ and $x(X) \in \mathcal{P}(M_2)$. Hence, $x(X) \in \mathcal{P}(M_1) \cap \mathcal{P}(M_2) = \mathcal{P}(M_1 \wedge M_2)$

Second, we show if $y \in \mathcal{P}(M_1 \wedge M_2)$ then there is an x such that $x \in Q(M_1 \wedge M_2)$ and $y = x(X)$. Since $y \in \mathcal{P}(M_1 \wedge M_2)$, hence, $y \in \mathcal{P}(M_1), \mathcal{P}(M_2)$. So, there exist x_1, x_2 such that $x_1(X) = x_2(X) = y$ and $x_1 \in Q(M_1), x_2 \in Q(M_2)$. Now, consider x obtained by stacking x_1, x_2 in coordinates X_1, X_2 and taking common coordinates corresponding to X. For this x, $x(X) = y$ and $x(X_1) \in Q(M_1), x(X_2) \in Q(M_2)$, hence $x \in Q(M_1 \wedge M_2)$. □

3.2 Extended Formulation of Matroid Union

In this section, we will state the main theorem we want to show, along with a formulation of the constraints for the matroid union polytope's extension.

Theorem 3.2 (Extension of Matroid Union)**.** *Given matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ which have a compact formulation, their matroid union polytope has a compact formulation.*

Let us denote the variables corresponding to elements in ground set of M_1 by x_e for $e \in S$ and similarly for M_2 by z_e for $e \in S$. Let us introduce some new variables v_e for $e \in S$. Let $\mathcal{P}(M_1)$'s compact extended formulation be given by $Q(M_1)$ and similarly $Q(M_2)$ for $\mathcal{P}(M_2)$ with variables of $Q(M_1), Q(M_2)$ being disjoint. Now, consider the polytope $Q(M_1 \vee M_2)$ given by

$$
\begin{aligned}
v_e &\geq 0 \ \forall \ e \in S \\
v_e &\leq 1 \ \forall \ e \in S \\
&Q(M_1) \\
&Q(M_2) \\
v_e &\leq x_e + z_e \ \forall \ e \in S
\end{aligned}
$$

Let the polytope corresponding to the union of matroids be denoted by $\mathcal{P}(M_1 \vee M_2)$ and let this be over the variables v_e for $e \in S$. The objective of the rest of the paper is to show that $Q(M_1 \vee M_2)$ is indeed an extended formulation of $\mathcal{P}(M_1 \vee M_2)$.

Lemma 3.3. *$Q(M_1 \vee M_2)$ is an extended formulation of $\mathcal{P}(M_1 \vee M_2)$.*

Let us define a projection map $v(.)$ which maps x to only coordinates corresponding to v_e for $e \in S$. Define similar maps $x(.), z(.)$ for M_1, M_2.

3.3 Characterization of Extension

In order to show the above theorem, we first start with a claim which is equivalent to theorem 3.2

Lemma 3.4. *Given a set $I \subseteq S$, I is independent in $M_1 \vee M_2$ if and only if there is a feasible point $p \in Q(M_1 \vee M_2)$ such that $v(p) = \chi_I$, where χ_I denotes the indicator vector corresponding to I in $\mathbb{R}^{|S|}$.*

Proof. For forward direction, let's take a $I \subseteq S$ with I being independent in $M_1 \vee M_2$. I will be formed by union of two independent sets one from M_1 and one from M_2. Let us denote them by I_1 and I_2 respectively. $\exists_{q_1}\ x(q_1) = \chi_{I_1}$ and $\exists_{q_2}\ z(q_2) = I_2$ with $v(p) = \chi_I$. We will choose v_e as 0 if x_e is zero in q_1 and z_e is zero in q_2 otherwise we will choose v_e as 1. This choices of v_e will satisfy the inequality $v_e \leq x_e + z_e\ \forall e \in S$. And since variables of q_1 and q_2 are disjoint , combining q_1 and q_2 along with the given assignment of v's will form a p such that $p \in Q(M_1 \vee M_2)$. □

Proving backward direction of lemma 3.4 will be done in the rest of paper.

Lemma 3.5. *Lemma 3.3 and Lemma 3.4 are equivalent.*

Proof. Assuming lemma 3.3, we will prove lemma 3.4. Let I be independent in $M_1 \vee M_2$. Then, $\chi_I \in \mathcal{P}(M_1 \vee M_2)$. Now, since $Q(M_1 \vee M_2)$ is an extended formulation, there exists a $p \in Q(M_1 \vee M_2)$ such that $v(p) = \chi_I$, showing the forward direction.
Now, to show the reverse direction of 3.4, let there exist a $p \in Q(M_1 \vee M_2)$ such that $v(p) = \chi_I$. Since $Q(M_1 \vee M_2)$ is an extended formulation, hence, $v(p) = \chi_I \in \mathcal{P}(M_1 \vee M_2)$. Now, $\mathcal{P}(M_1 \vee M_2)$ is the convex hull of indicators of independent sets of the matroid union. Any convex combination of indicators can give a vector with all integer coordinates if and only if the coefficient of convex combination is 1 for some indicator and zero for all others. Now, since $\chi_I \in \mathcal{P}(M_1 \vee M_2)$ is integral hence, it must be an indicator for an independent set.

Now, we assume lemma 3.4 to show lemma 3.3. First, let $x \in \mathcal{P}(M_1 \vee M_2)$. Then, x is a convex combination of some indicator vectors corresponding to independent sets. Let these vectors be $\chi_{I_1}, \cdots, \chi_{I_k}$, where $x = \sum_i \lambda_i \chi_{I_i}$. Now, by lemma 3.4 there are points $p_1, \cdots, p_k$ such that $v(p_i) = \chi_{I_i}$ and $p_i \in Q(M_1 \vee M_2)$. Then, let $p = \sum_i \lambda_i p_i$. Since $Q(M_1 \vee M_2)$ is convex, $p \in Q(M_1 \vee M_2)$ and $v(p) = \sum_i \lambda_i v(p_i) = x$.
Now, let $p \in Q(M_1 \vee M_2)$, we need to show $v(p) \in \mathcal{P}(M_1 \vee M_2)$. Let the corners of $Q'(M_1 \vee M_2)$ be denoted by p'_i for $i \in \{1 \cdots m\}$, where $Q'(M_1 \vee M_2)$ is the projection of $Q(M_1 \vee M_2)$ in the space spanned by x, z, v coordinates. Let $p' = \sum_i \lambda_i p'_i$, where $\sum_i \lambda_i = 1$ and p' is obtained by projecting p in the space spanned by x, z, v coordinates. Then, for all p'_i, assume $v(p'_i)$ is integral (shown in lemma 3.6). Now, since $v(p_i)$ is integral, hence, $v(p'_i) = v(p_i) = \chi_{I_i}$ for some I_i and $p_i \in Q(M_1 \vee M_2)$. Thus, using lemma 3.4, I_i is independent in $M_1 \vee M_2$ for all i, hence, χ_{I_i} are corners of $\mathcal{P}(M_1 \vee M_2)$. Now, $v(p) = v(p') = \sum_i \lambda_i v(p'_i) = \sum_i \lambda_i \chi_{I_i} \in \mathcal{P}(M_1 \vee M_2)$. □

In the above proof, the only remaining step is to show that corners of $Q'(M_1 \vee M_2)$ have integral projections in the v coordinates. Instead we show a stronger statement, that is the corners are integral in all coordinates.

Lemma 3.6. *All corners of $Q'(M_1 \vee M_2)$ are integral.*

Proof. Let p be a corner of the polytope. The polytope of $Q'(M_1 \vee M_2)$ is described by four types of constraints,

$$\begin{aligned} \sum_{i \in S} x_i &\leq r_1(S) \\ \sum_{j \in S'} z_i &\leq r_2(S') \\ v_i &\leq x_i + z_i \\ 0 \leq v_i &\leq 1 \end{aligned}$$

Since p is a feasible point in polytope, the rank constraints of both matroids are satisfied. Now consider the set of tight constraints for both matroids. Using proposition 2.5 we can find a chain of tight constraints, for both matroids. When these tight constraints are expressed as a system of linear equations $Ax = b$, all coordinates of b being integral. Since p is a corner in a polytope with (x, z, v) coordinates, we get $3n$ tight constraints where n is the size of the ground set of M_1, M_2. An example of the matrix A is shown in 9

$$A = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{9}$$

Since the tight constraints form a chain in both matroids, we can partition the constraints using gaussian elimination keeping the coordinates of b integral. In the example, this will result in a new matrix as shown in 10

$$A' = \left[\begin{array}{ccc|ccc|ccc} {\color{red}1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & {\color{red}1} & {\color{red}1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & {\color{blue}1} & {\color{blue}1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & {\color{blue}1} & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{10}$$

Further, in the columns corresponding to v coordinates, if both $v_i \leq x_i + z_i$ and $0 \leq v_1$ (corresp. $v_1 \leq 1$) are tight, the row with v_i having coordinate 1 can be added to the row with coordinate -1.

This operation in the example results in the matrix 11

$$A'' = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{11}$$

Now, it is enough to show that the determinant of A'' is either $1, -1$ or 0. Since p is a corner, the possibility of the determinant being 0 will be ruled out proving the claim.
Now, every column of v coordinates has atmost one 1 and all other rows contain zeros. Hence, expanding the determinant along these columns gives us the determinant of the submatrix with x, z coordinates in columns upto a sign. In the example the resulting matrix is given by 12

$$A''' = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \tag{12}$$

The resultant matrix has atmost two 1 entries per column, with number of rows and columns being equal. Hence, one of two things must be true by a simple averaging arguement

1. The matrix has exactly two ones in each row
2. The matrix has atmost one 1 in some row

In the second case we can expand the determinant of the matrix along the corresponding row if it contains a 1, and if not, the determinant is 0. Hence, eventually we either end up with a matrix corresponding to the first case or these inductive steps give an empty matrix in which case the determinant is +1 or −1. Hence, all that remains is to handle case 1.
In this case, since the matrix has exactly two ones in each row, it must have exactly two ones in each column. We know that the constraints corresponding to the rank constraints of the matroids when partitioned provide a single one per column. Hence, the other ones must arise from the other types of constraints. Let the rank constraints correspond to rows in the set S_1 and the other be from the set S_2. Then, the row operation $R_1 \to \sum_{i \in S_1} R_i - \sum_{j \in S_2} R_j$ yields zeros in the first row, showing that the determinant is 0. □

3.4 Extension of Partition Matroids

We first show extension complexity for partition matroid because of the simplicity of its polytope's characterization. Major task is to prove given $p \in Q(M_1 \cup M_2)$ with v_e integral, x_e and z_e non-integral, we can construct a $p^{'} \in Q(M_1 \cup M_2)$ such that $v(p) = v(p')$ and $x(p')$ and $z(p')$ are integral, hence proving backward direction of Lemma 3.4.

3.4.1 Construction of network flow

We will construct a network flow problem with suitable weights, such that if there exist $p \in Q(M_1 \cup M_2)$ with $v(p) = V$, then our network's max integral flow will give us p' with $v(p') = V$.

We will have 5 different layers in our network graph. First layer will have a single source node. Second layer will have nodes corresponding to 1 entries in V i.e, $\{\forall e \mid v_e = 1\}$ we will have a single node in the second layer. We will add edges between first and second layer of capacity 1 for all node pairs. Third layer will have nodes for x_e and z_e corresponding to 1 entries in V i.e, $\{\forall e \mid v_e = 1\}$. We will add edges of capacity 1 between second and third layer for nodes corresponding to same e. Fourth layer will have nodes corresponding to partitions in M_1 and M_2. x_e and z_e will be connected to their respective partition in which they are from third layer to fourth layer. Fifth layer just contains a single sink node. All the partition node will have a edge to sink with edge capacity equal to rank of partition.
For example, figure 1 shows the construction for network flow for two partition matroids M_1 and M_2. M_1's partition is $\{S_1, S_2\}$ with $S_1 = \{1\}$ and rank 1, $S_2 = \{2, 3\}$ and rank 1. M_2's partition is $\{S_3\}$ with $S_3 = \{1, 2, 3\}$ and rank 1, where $v_1 = v_2 = v_3 = 1$. Here, all edge capacities are hence, also equal to 1. Now, let's state the claim.

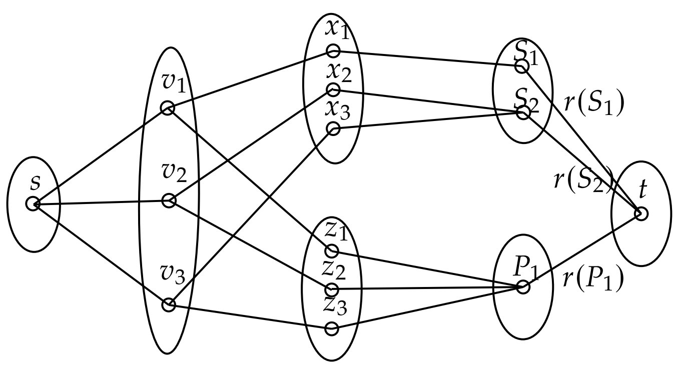

Figure 1: Example of a network flow obtained from a partition matroid

Lemma 3.7. *If there exists a $p \in Q(M_1 \vee M_2)$ with $v(p) = \chi_I$ for some $I \subseteq S$, then I is independent in $M_1 \vee M_2$ given both M_1, M_2 are partition matroids.*

Proof. Now, using the above construction for the given v, and partition matroids M_1, M_2, we get a network flow. Now, corresponding to a flow, we get a solution for $p' \in Q'(M_1 \vee M_2)$ as, $v_e = 1$ for all e, x_e, z_e are the values of incoming flow, and they must satisfy rank constraints by construction.

Now, the value of the flow, is equal to the max-flow. Hence, by ford-fulkerson algorithm, there must exist an integral max-flow. Thus, we get an integral solution for x, z, v. Now, as $v_i \leq x_i + z_i$, hence for all $i \in I$, $i \in I_1$ or $i \in I_2$, where I_1, I_2 are indicators of x, z respectively. Hence, there exist I_1, I_2 such that $I \subseteq I_1 \cup I_2$. Since, $I_1 \cup I_2$ is independent in $M_1 \vee M_2$, so is I. □

Observe that lemma 3.7 is the same as backward direction of lemma 3.4 for partition matroids. Hence, we get an important result.

Theorem 3.8. *If $M_1 = (S, I_1), M_2 = (S, I_2)$ are partition matroids with compact formulations, then $M_1 \vee M_2$ also has a compact formulation*

Proof. Application of lemmas 3.7, 3.4 and 3.5 shows that $Q(M_1 \vee M_2)$ is indeed an extended formulation of $\mathcal{P}(M_1 \vee M_2)$ and $Q(M_1 \vee M_2)$ is polynomial in $|S|$, hence, showing the result. □

3.5 Extension for General Matroids

Now that we have shown lemma 3.4 for partition matroids, we will use the techniques in its proof such as network flow in order to show lemma 3.4 for general matroids. Recall that the forward direction of the lemma has already been shown. For proving the backward direction, we will rely on two key lemmas. The first to obtain an integral feasible point in the proposed polytope given a feasible point, and the second showing that finding such an integral point shows that the given set is indeed an independent set in the matroid union.

3.5.1 Getting independent sets from an integral solution

To show that the given set is an independent set in the matroid union (given we have an integral feasible solution) we show the lemma,

Lemma 3.9. *If $\exists p$ such that $p \in Q(M_1 \vee M_2)$, $x(p)$ and $z(p)$ are integral and $v(p) = \chi_I$ then $I \in \mathcal{I}(M_1 \vee M_2)$.*

Proof. Choose I_1 as indicator vector of $x(p)$ i.e, $x(p) = \chi_{I_1}$ and I_2 as indicator vector of $z(p)$ i.e, $z(p) = \chi_{I_2}$. p satisfies $Q(M_1)$ implies $I_1 \in \mathcal{I}(M_1)$ and similarly $I_2 \in \mathcal{I}(M_2)$. $p \in Q(M_1 \vee M_2)$ implies $v_e \leq x_e + z_e \forall e$. This will imply either $x_e = 1$ or $z_e = 1$ if $v_e = 1$ since x_e, z_e and v_e all are integral. This implies $I \subseteq I_1 \vee I_2$ and since every subset of independent set is independent, $I \in \mathcal{I}(M_1 \vee M_2)$. □

3.5.2 Getting integral solution from a feasible solution

Here, we show how to obtain an integral solution from a feasible solution of the polytope $Q(M_1 \vee M_2)$. Coupling this with lemma 3.9, will finish the backward direction of lemma 3.4.

Lemma 3.10. *Given $I \subseteq S$, $p \in Q(M_1 \vee M_2)$ such that $v(p) = \chi_I$, then there exists $p' \in Q(M_1 \vee M_2)$ such that $x(p'), z(p')$ are integral and $v(p) = v(p') = \chi_I$.*

Proof. Given solution $p \in Q(M_1 \vee M_2)$ i.e, $x(p) \in \mathcal{P}(M_1)$ and $z(p) \in \mathcal{P}(M_2)$, we will show construction of $p' \in Q(M_1 \vee M_2)$ such that $x(p'), z(p')$ are integral and $v(p) = v(p') = \chi_I$. Let us denote $x(p')$ and $z(p')$ by x' and z' respectively. Initially, choose x' and z' as $x(p)$ and $z(p)$ respectively.

Modify x'_e and z'_e as zero if $v(p)_e = 0$. If $v(p)_e = 1$, then modify x_e and z_e such that $x'_e + z'_e = 1$ by reducing x_e or z_e. All constraints in $\mathcal{P}(M_1)$ and $\mathcal{P}(M_2)$ are of the form $\sum_e x_e \leq c$ and $\sum_e z_e \leq c$ by Edmond's characterization. This implies $x' \in \mathcal{P}(M_1)$ since we are only reducing values of x'_e in the above modification. Therefore, $\exists p'_1 \in Q(M_1)$ such that $x(p'_1) = x'$. Similarly, $\exists p'_2 \in Q(M_2)$ such that $z(p'_2) = z'$. Hence, $\exists$ a point $p' \in Q(M_1 \vee M_2)$ with $x(p') = x'$ and $z(p') = z'$. Then using 2.5 we can find a chain of inclusion wise maximal constraints for the tight sets corresponding to $x(p')$ in $\mathcal{P}(M_1)$, and a similar chain corresponding to $z(p')$. Let's denote these chains by $C^1 = \{C^1_i : i \geq 1\}$ and $C^2 = \{C^2_j : j \geq 1\}$. Now, the tight constraint corresponding to a chain can be given as,

$$\sum_{e \in C^1_i} x_e = rank(C^1_i)$$

Hence, this can be converted into a series of equalities obtained as,

$$\sum_{e \in C^1_i - C^1_{i-1}} x_e = rank(C^1_i) - rank(C^1_{i-1})$$

Similar constraints can be defined for C^2. Let the above system of equalities be denoted as C'. Now, this can be used to construct network flow similar to the construction used for the partition matroid. Augment flow by ε such that some slack constraints becomes tight in $\mathcal{P}(M_1)$ or $\mathcal{P}(M_2)$ or some x_e or z_e becomes integral (this can be done using lemma 3.11). Repeating the above process until all x_i and z_i become integral completes the proof. □

Lemma 3.11. *For a given network flow diagram of C', you will always be able to augment the flow by ε such that either some slack constraints became tight in $P(M_1)$ or $P(M_1)$ or some x_e or z_e becomes integral and tight constraints will remain tight.*

Proof. Consider a augmenting cycle in the network flow diagram which containing non-integral edges. If such a cycle doesn't exist then all x_e and z_e are already integral. There will exist a minimum ε which on augmenting to this cycle causes some non integral edges to become integral, let's denote this by α. There will exist a minimum ε which on augmenting to this path causes some slack constraint to become tight, let's denote it by β. Hence $\min(\alpha, \beta)$ is the required ε such that some slack constraints became tight in $\mathcal{P}(M_1)$ or $\mathcal{P}(M_2)$ or some x_e or z_e becomes integral. Note that tight constraints will remain tight during augmentation. This is because max flow implies that capacity constraints are tight for each partition. Since each element of the chain is a linear combination of elements of the partition, tightness of chain constraints is preserved. Moreover, any other tight constraint is a linear combination of the tight constraints in the chain, hence all tight constraints are preserved. □

3.5.3 Example of the integral point algorithm

The algorithm used in the proof of lemma 3.10 is iterative in nature, and uses the chaining of tight constraints heavily in order to construct a flow network. In this section, we will run this algorithm on the union of a 1-uniform and a 2-uniform matroid both of which have 3 elements in their ground

set.
Consider the matroids $M_1 = (S, I_1), M_2 = (S, I_2)$ with $S = \{1, 2, 3\}$. Consider $I \subseteq S$ as $I = \{1, 2, 3\}$.Let feasible point $p \in Q(M_1 \vee M_2)$ be such that $x(p) = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ and $z(p) = (\frac{3}{4}, \frac{1}{2}, \frac{3}{4})$. Then, we have the tight constraints,

$$x_1 + x_2 + x_3 \leq 1$$
$$z_1 + z_2 + z_3 \leq 1$$

So, we get the flow network 2. We see that in order to augment the flow, while respecting rank

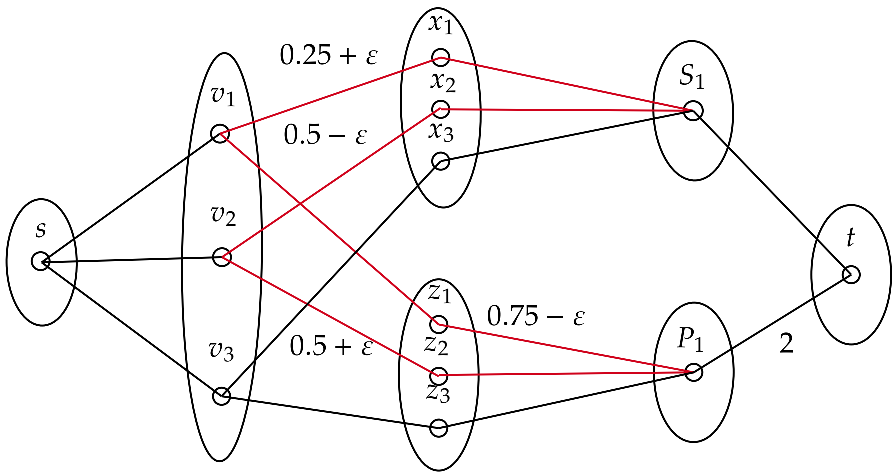

Figure 2: First iteration of algorithm

constraints, we can set $\varepsilon = 0.5$ to get $x = (0.75, 0, 0.25)$ and $z = (0.25, 1, 0.75)$. Hence, we get the new tight constraints as,

$$x_1 + x_3 \leq 1$$
$$x_1 + x_2 + x_3 \leq 1$$
$$z_1 + z_2 + z_3 \leq 2$$

Hence, paritioning gives us $x_1 + x_3 = 1, x_2 = 0$ as equalities in the flow network, corresponding to 3 Here, choosing $\varepsilon = 0.25$ does the trick, giving us the required integral solutions as $x = (1, 0, 0)$ and $z = (0, 1, 1)$. Observe that the union of the sets corresponding to these indicators indeed gives us I.

3.5.4 Finishing the proof

Now that we have shown lemmas 3.9, 3.10 we can show our main result.

Lemma 3.4. *Given a set $I \subseteq S$, I is independent in $M_1 \vee M_2$ if and only if there is a feasible point $p \in Q(M_1 \vee M_2)$ such that $v(p) = \chi_I$, where χ_I denotes the indicator vector corresponding to I in $\mathbb{R}^{|S|}$.*

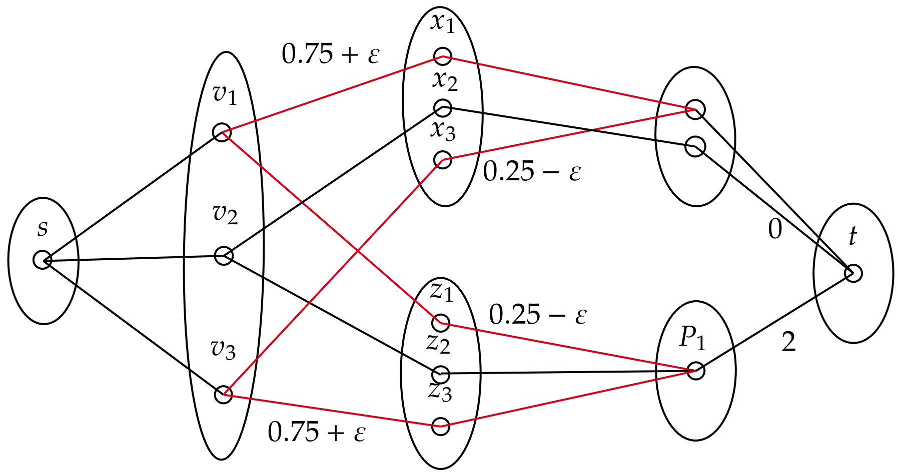

Figure 3: Second iteration of algorithm

Proof. The forward direction has already been shown.
For the backward direction, let $p \in Q(M_1 \vee M_2)$ such that $v(p) = \chi_I$. Then, from lemma 3.10, we have that there exists a point $p' \in Q(M_1 \vee M_2)$ such that $x(p'), z(p')$ are integral with $v(p') = \chi_I$. Now, using lemma 3.9 we have that $I \in \mathcal{I}(M_1 \vee M_2)$. □

Now that we have shown the critical lemma of this paper, we can use another lemma obtained earlier to show our proposed result.

Lemma 3.3. *$Q(M_1 \vee M_2)$ is an extended formulation of $\mathcal{P}(M_1 \vee M_2)$.*

Proof. Since lemma 3.5 holds, and lemma 3.4 has been showed, the proof follows. □

4 Discussion

Once we have the result that matroid union of matroids with compact formulations has a small extension complexity, it is tempting to use this result as a generator to get small extension for all matroids. But, union does not act as a generator for matroids, so such an approach can work for some matroids. After a brief review of the known extension complexities, we do not observe any novelties that can be derived from this result so far.

In the direction of finding small extension complexities for independent set polytopes, one of the major open questions is finding a small extension for linear matroids. We are also interested in conjectures given by Tony Huynh [Huy16] in his blog post. Both these open questions can serve as directions for future work.

5 Acknowledgements

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