## Logistics

6/1/25

4 weeks — Methods for Int. Optimization

10 weeks — Applications, Modelling, Solving (e.g. Decompositions)

- Vehicle Routing
- Line planning
- Network Design
- Multicommodity flows
- Crew Scheduling

(15-20%) Project (2 weeks) — Teams of 2/3

Quizzes

Midsem, Endsem

- No slides, material — just lectures

* AU if $\geq$ CC.

- Some references
  - CCZ (Conforti, Cornuéjols, Zambelli)
  - Walsey, 2nd Edition

} Int. Optimization

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## Int. Optimization

$$\min_{x} \; c^T x$$

$$\text{s.t.} \quad Ax \geq b$$

$$x \in \mathbb{Z}^n$$

$A_{m\times n}$

$c_{n\times 1}$

$b_{m\times 1}$

① Given $A, b$ find $x$ s.t. $Ax = b$ (GEM)

② Given $A, b$ find $x$ s.t. $Ax \geq b$

$$Ax - I\underbrace{s}_{\text{slack variables}} = b \quad , \quad s \geq 0$$

- Simplex method
  - ↳ Solve equality based system in every iteration.
- Interior point methods, Barrier method

(Problem)

Given $n$ objects of weights $w_1, \dots, w_n$ divide these $n$ objects into two sets of exactly equal weights, ensuring no object is left out (encode)

$$\sum x_i w_i = \sum w_i / 2 \quad , \quad x_i \geq 0, \; x_i \leq 1, \; x \in \mathbb{Z}^n$$

7/1

$\max \; c^T x$

$Ax \geq b$

$x_i \in \{0, 1\} \; \forall \; i$

Binary program

$\max \; c^T x$

$Ax \geq b$

$x \in \mathbb{Z}^n$

ILP

$\max \; c^T x + h^T y$

$Ax + Gy \leq b$

$x \geq 0$

$y \geq 0$

$x \in \mathbb{Z}^n, \; y \in \mathbb{R}^p$

MILP

feasible region $S = \{ (x, y) : Ax + Gy \leq b, \; x \geq 0, \; y \geq 0, \; x \in \mathbb{Z}^n, \; y \in \mathbb{R}^p \}$

# Branch and Bound

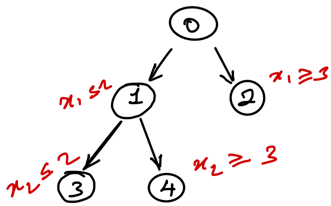
Hit and trial ← lower bound → try to bring closer.

drop ← relaxation ← upper bound → try to bring closer.

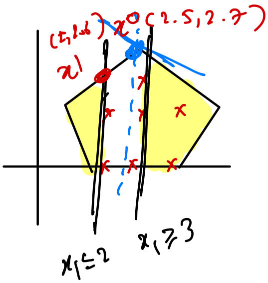
$x \in \mathbb{Z}^n$

(solve LP relaxation)

9/1




1. LP relaxation infeasible ✗
2. LP optimal soln $\in \mathbb{Z}^n$ ✓
3. LP has unbounded soln → ILP is unbounded / infeasible
4. LP opt $(x^0, y^0)$ with $x^0 \notin \mathbb{Z}^n$. Divide the problem into smaller parts and do branch – bound on each part




$x^0 = (1.3, 0.5, 1.7)$

$x_1 \le 1$ $\quad$ $x_1 \ge 2$ ← Extra Constraints

Prune by integrality, solve again !

$O(2^n)$ worst case

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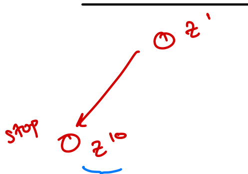
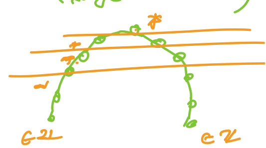


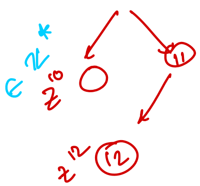

$x \in \mathbb{Z}^n, y \in \mathbb{R}^p$

if gap b/w $z^{10}$, $z^1$ is small, can stop !

$z^{10} \le z^* \le z^1$

# Can have large # levels even with 2 variables. (large # constraints)

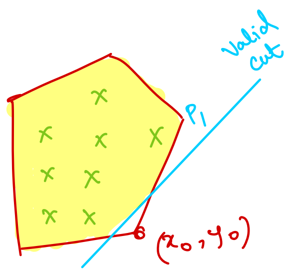



if $z^{12} \le z^{10}$ then prune node 12.

1. Best first search
2. Depth first search

## Cutting Plane




1. How to find cutting plane
2. How many # planes. ← exponentially many (worst case)

Practice :

Cutting ← Branch and bound on remaining

## Chvatal & Gomory

$\max \; x_1 + x_2$ $\quad \in \mathbb{Z}$

$\in \mathbb{Z} \Rightarrow$

$-5x_1 + x_2 + x_3 = 0$

$5x_1 + x_2 + x_4 = 5$

$x_1, x_2, x_3, x_4 \geq 0$

$\rightarrow$

$\bar{x} = (0, 0, 0, 5)$

set free vars to 0.

Simplex Method

$\downarrow$

Keep changing basis.

$\max \; 3 - \frac{3}{5}x_4 - \frac{2}{5}x_3$

$x_1 + \frac{1}{10}x_3 + \frac{1}{10}x_4 = \frac{1}{2}$

$\boxed{x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 = \frac{5}{2} \quad x \geq 0}$

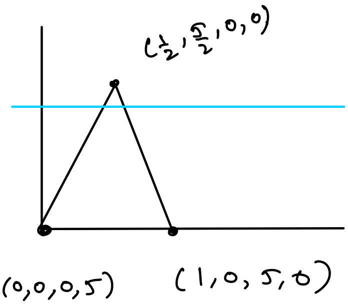
basic variables

$x_1$, $x_2$

$\bar{x} = (\frac{1}{2}, \frac{5}{2}, 0, 0)$

$x_3, x_4$ have $-$ve coeff, so this soln is best.

Look at this equation to get a cut




$\frac{1}{2}x_3 + \frac{1}{2}x_4 = \frac{1}{2} + \overbrace{2 - x_2}^{\in \{0,1,2\}}$

$\geq 0$

basic variable

$x \geq 0$

$\boxed{x_2 \leq 2}$

Cut proposed by Gomory, Chvatal

$x_1 - \frac{1}{10}x_3 + \frac{1}{10}x_4 = \frac{1}{2}$

$x_1 - x_3 + \frac{9}{10}x_3 + \frac{1}{10}x_4 = \frac{1}{2}$

$-x_1 + x_3 \geq 0$

$\boxed{x_3 \geq x_1}$ $\leftarrow$ Chops off the point

Given

$* \quad a_1x_1 + a_2x_2 + \cdots + a_nx_n = a_0$

$x_1 \ldots x_n \geq 0$ $\Big\} R$

Then, $\sum_{j=1}^{n}(a_j - \lfloor a_j \rfloor)x_j \geq a_0 - \lfloor a_0 \rfloor$ is satisfied by all $x \in \mathbb{Z}^n \cap R$

$\sum_{j=1}^{n} a_j x_j = a_0 \Rightarrow \left\lfloor \sum_{j=1}^{n} a_j x_j \right\rfloor = \lfloor a_0 \rfloor$ $\Rightarrow \sum_{j=1}^{n} \lfloor a_j \rfloor x_j \leq \lfloor a_0 \rfloor \; (*)$

$\geq$

$(\because x \geq 0)$ $\quad \sum_{j=1}^{n} \lfloor a_j x_j \rfloor$ $\quad x_j \in \mathbb{Z}$

$\geq$

$\lfloor a_j \rfloor x_j$

$R - (*)$

$\Rightarrow \sum_j a_j - \lfloor a_j \rfloor x_j \geq a_0 - \lfloor a_0 \rfloor$

If in std form $\quad x_{b1} + \cdot a_i x_i + \cdots = b_1$ $\quad$ CG procedure.

0 for smplx pt. $= (a_2 - \lfloor a_2 \rfloor)x_2 + \cdots \geq b_1 - \lfloor b_1 \rfloor$ say 0.6 $\leftarrow$ current pt chopped off

zeros.

Max $c^T x$

$x^T A_1 x \leq b_1$

$\vdots$

$x^T A_n x \leq b_n$

$x \in \mathbb{Z}^n$

Quadratic Constr. IP

(undecidable)

$Ay = b$, $y \geq 0$ infeasible $\iff$ $A^T p \geq 0$, $b^T p < 0$

21 Jan — convex hull lec (Ask later)

Knapsack

Given a bag of capacity $b$ and $n$ types of items that can be taken in bag with $a_i$ being the weight of one piece of item $i$ and $c_i$ its value. How many of each item will you pack in the bag to maximize total value?

$$\max \quad c_1 x_1 + \cdots + c_n x_n \qquad \text{(Knapsack model)}$$

Knapsack constraint

$$a_1 x_1 + \cdots + a_n x_n \leq b \quad \text{(budget)}$$

$$x \geq 0$$

$$x \in \mathbb{Z}^n$$

E.g. $21x_1 + 17x_2 \leq 40$ $R_1$

$x_1 \leq 1$ $R_2$

$R_2 \subset R_1$

Binary Knapsack $\quad x \in \{0,1\}^n$

Bin packing Given $n$ items of weights $w_1, w_2, \ldots, w_n$ and identical boxes of capacity '$b$' each. How many boxes are required at minimum to pack all items.

$$z_j = \begin{cases} 1, & \text{if box } j \text{ is used} \\ 0, & \text{o/w} \end{cases} \qquad j \in [n]$$

$$x_{ij} = \begin{cases} 1, & \text{if item } i \text{ put in box } j \\ 0 \end{cases} \qquad i \in [n],\ j \in [n]$$

$$\min \sum_{i=1}^{n} z_i$$

$$\text{s.t.} \quad \sum_{i=1}^{n} w_i x_{ij} \le b \quad ; \; j \in [n]$$

$$x_{ij} \le z_j \quad ; \; i, j \in [n]$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad ; \; i \in [n]$$

Cutting stock

$$x_j = \begin{cases} 1, & \text{if roll } j \text{ is used} \\ 0, & \text{if " — not "} \end{cases}$$

$D = \sum d_i$

# small rolls of width $w_i$

$z_{ij}$ = No. of rolls of width $i$ that we cut from roll $j$

$$\min \quad x_1 + \cdots + x_D$$

$$\text{s.t.} \quad \sum_j z_{ij} w_j \le \underbrace{W_j \; x_j}_{} \quad \boxed{\text{Imp}}$$

Coefficient tightening

$$\sum_j z_{ij} \ge d_i$$

$$z_{ij} \le \frac{W}{w_i} x_j$$

$$x_j \in \{0, 1\}$$

saw

$\sum a_i x_i \le b \, y_i$

if $y_i = 0$, $x_i = 0$

$y_i = 1$ (ok)

$x_i \le y_i$

eliminates some fractional soln

$y_i \in \{0, 1\}$

\# cons $= N + nN + n$

\# vars $= N + nN$

Large formulation for Cutting Stock (ref pg 50: conforti) ← solve inc bigger LPs (add cols successively)

solve dual

Feasible patterns $\quad P_1 : 3 \times 30$

$P_2 : 2 \times 30 + 1 \times 25$ - - - -

$x_1$

$x_2$ = # times $P_2$ is used

Naive lower bound : $\dfrac{\sum w_i d_i}{W}$

$$\min \quad \sum_s x_s$$

$$(3x_1 + 2x_2 + \cdots) \ge d_{30}$$

$$0 x_1 + \cdots$$

$$x_s \in \mathbb{N}$$

Tip: Write "maximal" patterns

* Every column is a pattern

Alternate objective : min # patterns used.

Tighter relaxation

## Packing Nodes in Graph

(Packing = independent set)

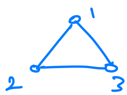
$$x_i = \begin{cases} 1 & ; \text{ node } i \text{ is selected} \\ 0, & \text{else} \end{cases}$$

$$\text{Max} \quad \sum_{i=1}^{n} w_i x_i$$

$$\text{s.t.} \quad x_u + x_v \le 1 \quad \forall \{u,v\} \in E$$

$$x \in \{0,1\}^n$$

Some tricks,




$$x_1 + x_2 + x_3 \le 1 \quad \leftarrow \text{tighter LP formulation}$$

$$0 \le x \le 1$$

## Covering Nodes of a Graph

Select a few nodes s.t. all nodes are either selected or are neighbors of selected nodes.

$$\text{Min} \quad \sum_{i=1}^{n} w_i x_i$$

$$x_v + \sum_{u \in N(v)} x_u \ge 1 \quad \forall u \in V$$

$$x \in \{0,1\}^n$$

## Graph Partitioning

e.g. $$\left.\begin{array}{l} \min \sum w_i x_i \\ x_i + x_j = 1 \quad \forall (u,v) \in E \\ x \in \{0,1\}^n \end{array}\right\} \text{bipartite components.}$$

**Min-Cut**

$$\min \sum w_e z_e$$

$$x_s = 1, \; x_t = 0$$

$$z_e \ge x_u - x_v$$

$$x_v - x_u$$

$$x \in \{0,1\}^n$$

Project → Max-cut (pg 56–57)

→ Covering of Steiner triples (2.4.6)

**Set-cover**

$S_1, S_2, \ldots, S_m$

$(a_1, a_2, a_3)$ . . . . .

each pair unique $S$.

pick elements covering all $S_i$ $(e \cap S_i \ne \emptyset)$

## TSP (Complete graph HAM-cycle minimum weight)

not allowed

### (DFJ Formulation)

↓

exp many constraints

$x_{ij}$ : $(i,j)$ is part of tour

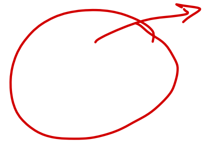
$$\sum_{j=1}^{n} x_{ij} = 1, \quad \sum_{j=1}^{n} x_{ji} = 1$$

$$x_{ij} \geq 0$$

$$\min \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} x_{ij} w_{ij}$$

**Connectedness**

$S \subset V$



at least one outgoing edge

$$\sum_{i \in S} \sum_{j \in \bar{S}} x_{ij} \geq 1$$

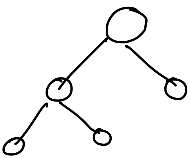
$$\forall S \subset V, |S| \geq 2$$

**Idea :**

1. Solve without S
2. If S violated, add constraint and solve model again

(Works in practice)

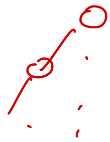
**Idea :**



B & B tree for vanilla constraints

→

$S_1$

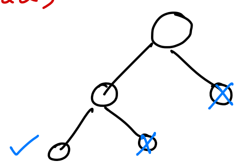


→

$S_1, S_2$

. . .

Instead,



(open leaf nodes)

add $S_1$ to all feasible leaves and continue, instead of throwing out the previous tree.

(lazy cuts)

### MTZ formulation

$$x_{ij} = \begin{cases} 1, & \text{if we go from node } i \text{ to } j \text{ in tour} \\ 0, & \text{o/w} \end{cases}$$

$u_i$ = position of node $i$ in tour. $\in \{1 \dots n\}$

$$\min \sum_{i \neq j} c_{ij} x_{ij}, \quad \sum_{j=1}^{n} x_{ij} = 1, \quad \sum_{j=1}^{n} x_{ji} = 1$$

$i$ to $j \Rightarrow u_j \geq u_i + 1$

- All $u_i$s are different

$$u_i - u_j + 1 \leq n(1 - x_{ij}) \quad (i, j \neq 1)$$

$$1 \leq u_i \leq n$$

$x_{ij} = 0 \Rightarrow u_i - u_j \leq n-1$

$x_{ij} = 1 \Rightarrow u_j \geq u_i + 1$

$x \in \{0,1\}^E$
(slower in B & B)

If $\exists$ subtour not cont node 1

$$\sum_{(i,j) \in C} (u_i - u_j + 1 \leq n(1 - x_{ij}))$$

$|C| \leq 0$. (contradiction).

## 3-index formulation

$x_{i,j,k} = 1$ if we go from $i$ to $j$ in $k^{th}$ leg of tour

$$\min \sum_{k} \sum_{i,j} x_{ij,k} c_{ij}$$

$$\sum_{j=1}^{n} \sum_{k} x_{ijk} = 1 \quad , i \in [n]$$

$$\sum_{i=1}^{n} \sum_{k} x_{ijk} = 1 \quad , j \in [n]$$

→ enter/exit city $j$ exactly once

$$\sum_{i \neq j} x_{ijk} = 1 \quad , k \in [n]$$ → every step move somewhere.

$$\underbrace{\sum_{i=1}^{n} x_{ijk}}_{\text{entering } j \text{ at turn } k} = \sum_{\substack{q=1 \\ q \neq j}}^{n} x_{jq,k+1} \quad \forall j, k$$

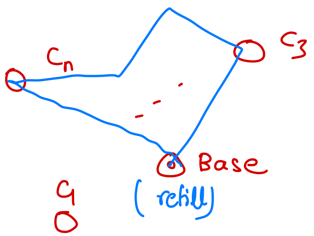
Enter at turn $k \Rightarrow$ leave at turn $k+1$

$$\sum_{i=2}^{n} x_{i1n} = 1$$ → reach node 1 in the last turn

$$\sum_{i=2}^{n} x_{1i1} = 1$$ → leave node 1 in $1^{st}$ turn.

## Capacitated Vehicle Routing Problem (CVRP)

{Toth & Vigo — ref book}




e.g.

Capacity of truck = 40 units

Inputs : Base : ( 0 )

Customers : $1 \ldots n$

Demands : $q_1 \cdots q_n$

Capacity of vehicle : $Q$

Cost of moving b/w $i$ and $j$ : $C_{ij}$

Number of vehicles : $K$

Bin packing + TSP

Q1 Pack $n$ customers in $k$ bins each of size $Q$

Q2 How to sequence each bin / route

Obj : Minimize cost of movements (without splits : all demand met at same time)

$$x_{i,j} = \begin{cases} 1 & \text{if vehicle goes from node } i \text{ to } j \text{ in solution} \\ 0 & \end{cases}$$

$$\min \sum_{\substack{i,j \\ i \neq j}} C_{ij} \, x_{ij}$$

depart once

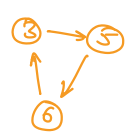
enter once

$$\text{s.t.} \quad \sum_{\substack{j=1 \\ j \neq j}}^{n} x_{ij} = \sum_{\substack{i=1 \\ i \neq j}}^{n} x_{ij} = 1 \qquad i, j \in \{1 \cdots n\}$$

(C−I) (C−II)

$$\sum_{j=1}^{n} x_{0j} \leq K$$

want to avoid

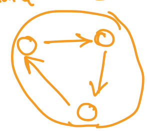


Laporte, Nobert, Desrochers

Take any subset $S$ $\quad \{n_1, \dots, n_k\} \leftarrow$ total demand $= D$ then need $\left\lceil \frac{D}{K} \right\rceil$ trucks atleast.

Let $r(S) :=$ minimum number of trucks required to meet demand of $S$.

violated $\Leftarrow S$

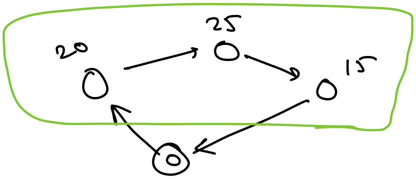


$\left\lceil \frac{q(S)}{Q} \right\rceil \leftarrow$ lower bound for $r(S)$ : actual model from bin packing

$$\sum_{i \notin S} \sum_{j \in S} x_{i,j} \geq \left\lceil \frac{q(S)}{Q} \right\rceil \quad \forall S \subseteq \{1 \dots n\}$$

— Eliminates subtours + takes care of capacity constraints.

Let $\exists$ tour where capacity is violated




VRP w/ fewer constraints ? Yes.

## MTZ formulation

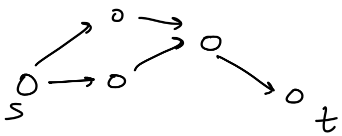
$$\min \sum c_{ij} x_{ij}$$

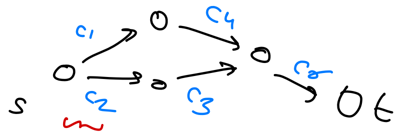
s.t. C − I, II

Let $u_i =$ demand already distributed by the vehicle at the time of enteri

## 2.10 Network Design

**Network flow**




- costs per unit flow given.
- Some capacities given on the edges.
- Some quantities are moved

$y_{ij}$ = flow along $(i,j)$

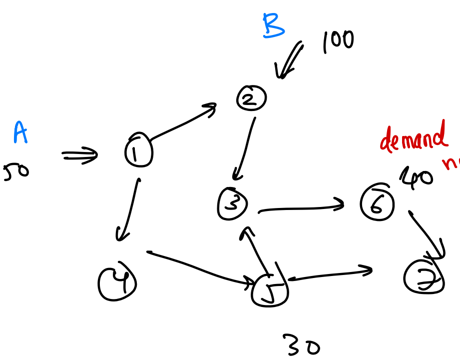
$$\sum_{i \in V} y_{ij} = \sum_{k \in V} y_{jk} \quad \forall j \in V - \{s,t\}$$

$$\sum_{k \in V} y_{s,k} = I + \sum_{i \in V} y_{i,s}$$

$$\sum_{i \in V} y_{i,t} = I + \sum_{k \in V} y_{t,k}$$

$$\underbrace{u_{ij}}_{\text{capacity of } (i,j)} \geq y_{ij} \geq 0$$

Multicommodity flow




demand on nodes (of each product)

* Write LP for each commodity

$y_{ij}^k$ = qty of commodity $k$ travelling on arc $(i,j)$

$$\sum_{j \in V} y_{i,j}^k - \sum_{l \in V} y_{l,i}^k = \begin{cases} s_i^k & \text{if } i \text{ is supply node for comm } k \\ -d_i^k & \text{if } i \text{ is demand " — "} \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in V.$$

$$\sum_k y_{i,j}^k \leq u_{i,j}, \quad y_{i,j}^k \geq 0 \qquad \forall (i,j) \in E$$

## Flow without splits (Hard problem)

$g^k$ = demand for commodity $k$

now we want path for each demand

$$y_{i,j}^k = \text{whether } (i,j) \in A \text{ is carrying commodity } k$$

Assumption: single demand, supply node per commodity.

$$\sum_k y_{i,j}^k - \sum_l y_{l,i}^k = \begin{cases} \text{supply}/g^k & (+1) \\ -\text{demand}/g^k & (-1) \\ 0 & \end{cases}$$

$$\sum_k (y_{i,j} * g^k) \le u_{i,j}$$

$$\min \sum_{i,j,k} c_{ij}^k \, y_{ij}^k$$

* Network design has additional fixed cost for connecting an edge.

## Quadratic Assignment

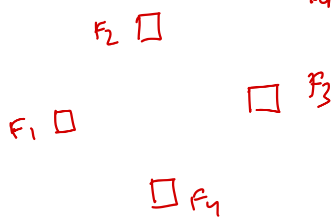
facility loc (1... 4 can be kept in any permutation)

Location : 1 . . . n

$d_{ij}$ = distance b/w fac. locations

Facilities = 1 . . . n

$f_{ij}$ = flow b/w facilities.




$$x_{ij} = \begin{cases} 1, & i^{th} \text{ facility at } j^{th} \text{ location} \\ 0, & o/w \end{cases}$$

$$\sum_j x_{ij} = 1 \quad \forall i$$

$$\sum_i x_{ij} = 1 \quad \forall j$$

$$x \in \{0,1\}^{n \times n}$$

$$\min \sum_{i,j,p,q} \underbrace{x_{ij} \, x_{pq}}_{\text{quadratic program.}} \, f_{ip} \, d_{jq}$$

Quadratic assignment (20s / 30s) locations. (TSP 100 k cities)

↳ poor LP relaxation.

LP - relaxation

$$y_{i,j,p,q} \leq x_{i,j}$$

$$y_{i,j,p,q} \leq x_{p,q}$$

$$y_{i,j,p,q} \geq x_{ij} + x_{pq} - 1$$

Replace $x_{ij} \, x_{pq}$ with $y_{ijpq}$ .