# Random Walks and forbidden minors I

Ameya Singh, Krishna Agaram & Soham Joshi

Prof. Akash Kumar

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# Problem Statement

**Theorem**

If $G$ is $\epsilon$-far from being $H$-minor-free, then there exists an algorithm finds an $H$-minor of $G$ with probability at least $2/3$. Furthermore, the algorithm has a running time of $dn^{1/2+O(\delta r^2)} + d\epsilon^{-2\exp(2/\delta)/\delta}$

# Algorithm

The main procedure of the paper FindMinor($G, \epsilon, H$) tries to find an $H$-minor in $G$.

1. LocalSearch($s$) : This procedure performs a small number of short random walks to find pieces of small conductance
2. FindPath($u, v, k, i$) : This procedure tries to find a path from $u$ to $v$.
    1. $i$ : length of walk
    2. $k$ : number of walks from $u$ to $v$
3. FindBiClique($s$) : Attempts to find sufficiently large biclique minor. It
    1. Generates seed sets $A$, $B$ using random walks from $s$.
    2. Calls FindPath on all pairs in $A \times B$.

# Parameters

1. $\delta$ : An arbitrary small constant
2. $r$ : The number of vertices in $H$
3. $\ell$ : The random walk length. Set as $n^{5\delta}$
4. $\epsilon_{\text{CUTOFF}} = n^{\frac{-\delta}{\exp(2/\delta)}}$
5. KKR($F, H$) : Exact $H$-minor finding procedure by Kawarabayashi, Kobayashi and Reed. Running time $= O(V^2)$

# Subroutines

$\texttt{FindMinor}(G, \varepsilon, H)$

1. If $\varepsilon < \varepsilon_{\mathsf{CUTOFF}}$, query all of $G$, and output $\texttt{KKR}(G, H)$
2. Else
   (a) Repeat $\varepsilon^{-2} n^{35\delta r^2}$ times:
      i. Pick u.a.r. $s \in V$
      ii. Call $\texttt{LocalSearch}(s)$ and $\texttt{FindBiclique}(s)$.

$\texttt{LocalSearch}(s)$

1. Initialize set $B = \emptyset$.
2. For $h = 1, \ldots, n^{7\delta r^2}$:
   (a) Perform $\varepsilon^{-1} n^{30\delta r^2}$ independent random walks of length $h$ from $s$. Add all destination vertices to $B$.
3. Determine $G[B]$, the subgraph induced by $B$.
4. Run $\texttt{KKR}(G[B], H)$. If it returns an $H$-minor, output that and terminate.

# Subroutines

`FindBiclique`$(s)$

1. For $i = 5r^2, \dots, 1/\delta + 4$:
   (a) Perform $2r$ independent random walks of length $2^{i+1}\ell$ from $s$. Let the destinations of the first $r$ walks be multiset $A$, and let the destinations of the remaining walks be $B$.
   (b) For each $a \in A$, $b \in B$:
      i. Run `FindPath`$(a, b, n^{\delta(i+18)/2}, i)$.
   (c) If all calls to `FindPath` return a path, then let the collection of paths be the subgraph $F$. Run `KKR`$(F, H)$. If it returns an $H$-minor, output that and terminate.

`FindPath`$(u, v, k, i)$

1. Perform $k$ random walks of length $2^i\ell$ from $u$ and $v$.
2. If walks from $u$ and $v$ terminate at the same vertex, return these paths. (Otherwise, return nothing.)

# $R$-Returning probability vectors

## Definition

For any set of vertices $R$, $s \in R, u \in R$,and $i \in \mathbb{N}$ ,we define the $R$-returning probability as follows.We denote by $q^{(i)}_{[R],s}(u)$ the probability that a $2^i\ell$-length random walk from $s$ ends at $u$,and encounters a vertex in $R$ at every $j\ell$th step for all $1 \leq j \leq 2^i$. The $R$-returning probability vector,denoted by $q^{(i)}_{[R],s}$ is the $|R|$ -dimensional vector of returning probabilities.

# Some more lemmas

**Lemma**

$q^{(i+1)}_{[R],s}(u) = q^{(i)}_{[R],s} . q^{(i)}_{[R],u}$

This follows from symmetry of the random walk matrix.

**Lemma**

$q^{(i)}_{[R],s} = (\mathbb{P}^R_R M^\ell \mathbb{P}_R)^{2^i} 1_s$

This is because the matrix $P_R$ filters $R$ from $V(G)$.

**Lemma**

$|R|^{-1} \sum_{s \in R} \|q^{(i)}_{[R],s}\|_1 \geq (|R|/n)^{2^i}$

**Proof.**

The case $i = 0$ follows from Cauchy-Schwarz. The general case then follows by the power-mean inequality. □

# Stratification

Stratification results in a collection of disjoint sets of vertices denoted by $S_0, S_1, \cdots$ which are called strata. The corresponding residue sets are denoted by $R_0, R_1, \cdots$. The zeroth residue $R_0$ is initialized before stratification, and subsequent residues are defined by the recurrence $R_i = R_0 \setminus \bigcup_{j<i} S_j$.

## Definition

Suppose $R_i$ has been constructed. A vertex $s \in R_i$ is placed in $S_i$ if $\|q^{(i)}_{[R],s}\|_2^2 \geq 1/n^{\delta i}$

# Further claims

## Claim

For all $s \in R_i$ and $1 \leq j \leq i$, $\left\| q^{(j)}_{[R_i],s} \right\|_2^2 \leq \frac{1}{n^{\delta(j-1)}}$.

If not (for some $j$), $\left\| q^{(j)}_{[R_{j-1}],s} \right\|_2^2 \geq \left\| q^{(j)}_{[R_i],s} \right\|_2^2 > \frac{1}{n^{\delta(j-1)}}$, implying $s \in S_{j-1}$ (and so $s \notin R_i$, contradiction).

# Lower-bounding the 1-norm

## CLaim

For each $s \in R_i$ and $2 \leq j \leq i+1$, $\left\| q^{(j)}_{[R_i],s} \right\|_\infty \leq \frac{1}{n^{\delta(j-2)}}$.

This follows from $q^{(j+1)}_{[R_i],s} = q^{(j)}_{[R_i],s} \cdot q^{(j)}_{[R_i],s}$ and Cauchy-Schwarz.

## Claim

For all $s \in S_i$, $\left\| q^{(i+1)}_{[R_i],s} \right\|_1 \geq \frac{1}{n^\delta}$.

## Proof

The key point is using Hölder to conclude $\left\| q^{(i+1)}_{[R_i],s} \right\|_2^2 \leq \left\| q^{(i+1)}_{[R_i],s} \right\|_1 \left\| q^{(i+1)}_{[R_i],s} \right\|_\infty$. Using the above bounds on the two other terms yields the result.

# Most vertices in early strata

**Lemma**

Suppose $\epsilon > \epsilon_{\text{CUTOFF}}$. At most $\epsilon n / \log(n)$ vertices are in $R_{1/\delta+3}$

# Correlation Lemma

## Lemma

Fix $s \in R_i$. Then the following holds:

$$\mathbb{E}_{u_1, u_2 \in \mathcal{D}_{s,i} \times \mathcal{D}_{s,i}} \left[ q^{(i)}_{[R_i], u_1} \cdot q^{(i)}_{[R_i], u_2} \right] \geq \frac{1}{\left\| q^{(i+1)}_{[R_i], s} \right\|_1^2} \frac{\left\| q^{(i+1)}_{[R_i], s} \right\|_2^4}{\left\| q^{(i)}_{[R_i], s} \right\|_2^2}$$

## Proof

Extremely clever manipulation and Cauchy-Schwarz.