Insights into the Core of the Assignment Game

A CS 602 Presentation

Ashwin Abraham
Harshit Morj
Soham Joshi

IIT Bombay

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Definition (Assignment Game)

The Assignment game is a weighted bipartite graph $G = (U, V, E)$ with $w : E \rightarrow \mathbb{R}^+$ being an assignment of weights to the edges.

One can think of this game in the following setting: Consider a mixed doubles tennis tournament, with U being the set of female players and V being the set of male players. An edge $(u, v) \in E$ iff $u \in U$ and $v \in V$ can pair up to form a team and $w(u, v)$ represents the expected earnings if (u, v) pair up. The total worth of a game is the weight of a maximum weight matching of G.
Given a maximum weight matching, we wish to distribute the profits in such a way that no subset of the players will be better off by seceding.

Preliminaries and Definitions

Definition (The Grand Coalition and coalitions)

The set of all players $U \cup V$ is called the Grand Coalition. Any subset of players $S_u \cup S_v$ where $S_u \subseteq U$ and $S_v \subseteq V$ is called a coalition.

Definition (The Characteristic Function)

The worth of a coalition $S_u \cup S_v$ is the weight of a maximum weight matching in the subgraph induced by $S_u \cup S_v$. The worth of the game is defined as the worth of the grand coalition. The characteristic function of the game $p : 2^U \times 2^V \to \mathbb{R}^+$ is the function such that $p(S_u \cup S_v)$ is the worth of the coalition S_u, S_v.

Preliminaries and Definitions

Definition (Imputation)

An imputation gives us a way of splitting the worth of a game among the players. It consists of a pair of functions $u : U \to \mathbb{R}^+ \cup \{0\}$ and $v : V \to \mathbb{R}^+ \cup \{0\}$ such that $\sum_{i \in U} u(i) + \sum_{j \in V} v(j) = p(U \cup V)$

Definition (The Core of the Assignment Game)

An imputation (u, v) is in the core of the assignment game iff, for every coalition $S = S_u \cup S_v$, we have $\sum_{i \in S_u} u(i) + \sum_{j \in S_v} v(j) \geq p(S)$

Linear Programming for the Assignment Game

The integer program for the maximum matching problem is:

$$\max \sum_{(i,j)\in E} w_{ij}x_{ij}$$
$$\texttt{st} \sum_{j\in V} x_{ij} \leq 1, \forall i \in U$$
$$\sum_{i\in U} x_{ij} \leq 1, \forall j \in V$$
$$x_{ij} \in \{0,1\}, \forall (i,j) \in E$$

Relaxing this to a Linear Program only changes the last constraint to $x_{ij} \geq 0$ for all $(i,j) \in E$. It can be shown that there exists an optimal solution for this Linear Programming that has $x_{ij} \in \{0,1\}$ for all $(i,j) \in E$, ie the solution of the relaxed LP is also a solution to the IP[1].

[1] This is because it can be shown that every $k \times k$ submatrix of the coefficient matrix has determinant 0 or ± 1

The Dual Problem

The Dual Problem for the relaxed LP is:

$$\min \sum_{i \in U} u_i + \sum_{j \in V} v_j$$

$$\texttt{st } u_i + v_j \geq w_{ij}, \forall (i,j) \in E$$

$$u_i \geq 0, \forall i \in U$$

$$v_j \geq 0, \forall j \in V$$

Theorem (Shapley, Shubik)

A vector (u, v) is an imputation in the core of the assignment game iff it is an optimal solution to the dual LP shown above.

Corollary

The core of the assignment game is non-empty.

Proof of the Shapley-Shubik Theorem

If (u, v) is an optimal solution to the dual LP then we have $u_i \geq 0, \forall i \in U$, $v_j \geq 0, \forall j \in V$. By strong duality the solution to the dual LP is equal to that of the primal LP, which as we have shown, is the weight of a maximum weight matching, ie $p(U \cup V)$. Therefore,

$$\sum_{i \in U} u_i + \sum_{j \in V} v_j = p(U \cup V)$$

and hence (u, v) is an imputation. Since $u_i + v_j \geq w_{ij}, \forall (i, j) \in E$, for any matching in $S_u \cup S_v$, we have $\sum_{i \in S_u} u_i + \sum_{j \in S_v} v_j$ being at least the weight of the matching and hence $\sum_{i \in S_u} u_i + \sum_{j \in S_v} v_j \geq p(S_u \cup S_v)$ - ie (u, v) is in the core of the assignment game.

Proof of the Shapley-Shubik Theorem

If (u, v) is an imputation then we have $u_i \geq 0$ for all $i \in U$ and $v_j \geq 0$ for all $j \in V$. Note that we also have $\sum_{i \in S_u} u_i + \sum_{j \in S_v} v_j = p(U \cup V)$ which is the the weight of a maximum weight matching in G, which as we have shown is the optimal solution for the dual LP. Since (u, v) is in the core of the assignment game, $\sum_{i \in S_u} u_i + \sum_{j \in S_v} v_j \geq p(S_u \cup S_v)$ for any $S_u \subseteq U$ and $S_v \subseteq V$. Taking $S_u = \{i\}$ and $S_v = \{j\}$ for some $(i, j) \in E$, we get $u_i + v_j \geq w_{ij}$, and hence (u, v) satisfies all the constraints of the dual LP and also attains the optimal value, making it an optimal solution.

The General Graph Matching Game

Here, we generalize the assignment game to graphs that need not be bipartite. Given a graph $G = (V, E)$ and positive weights on the edges according to the function $w : E \to \mathbb{R}^+$. The worth $p(S)$ of a subset S of vertices, is the weight of a maximum weight matching in the subgraph induced by S. This game may have an empty core. To deal with the possibility that the core may be empty, two different approaches have been suggested.

The General Graph Matching Game

The first is that of the *least core*.

Definition (Least Core)

An imputation is in the least core iff it maximizes $\min_{S \subseteq V} v(S) - p(S)$ given $v(V) = p(V)$ and $v(\emptyset) = 0$.

This involves exponentially many constraints but if an efficient separation oracle is devised, it can be solved in polynomial time by the oracle method.

The General Graph Matching Game

The other is that of an *approximate core*:

Definition (Approximate Core)

An imputation is in the approximate core if for every $S \subseteq V$, $v(S) \geq \frac{2}{3}p(S)$

Definitions

Definition (Mixed doubles team)

By a *mixed doubles team* we mean an edge in G; a generic one will be denoted as $e = (u, v)$. We will say that e is :

1. *essential* if e is matched in every maximum weight matching in G.
2. *viable* if there is a maximum weight matching M such that $e \in M$, and another, $M^{'}$ such that $e \notin M^{'}$.
3. *subpar* if for every maximum weight matching M in G, $e \notin M$.

More Definitions

Definition

Let y be an imputation in the core of the game. We will say that e is *fairly paid* in y if $y_u + y_v = w_e$ and it is *overpaid* if $y_u + y_v > w_e$. Finally, we will say that e is *always paid fairly* if it is fairly paid in every imputation in the core.

More Definitions

Definition

A generic player in $U \cup V$ will be denoted by q. We will say that q is:

1. *essential* if q is matched in every maximum weight matching in G.
2. *viable* if there is a maximum weight matching M such that q is matched in M and another, M' such that q is not matched in M'.
3. *subpar* if for every maximum weight matching M in G, q is not matched in M.

Vazirani's Theorem

Theorem (Vazirani)

The following hold :

1. *For every team* $e \in E$*:*
$$e \text{ is always paid fairly} \iff e \text{ is viable or essential}$$
2. *For every player* $q \in (U \cup V)$*:*
$$q \text{ is paid sometimes} \iff q \text{ is essential}$$

(Useful) Consequences of Vazirani's Theorem

1. Negating both sides of the first statement proved in Theorem 2 we get the following double implication. For every team $e \in E$:

$$e \text{ is subpar } \iff e \text{ is sometimes overpaid}$$

   giving rise to a strange observation, *Whereas viable and essential teams are always paid fairly, subpar teams are sometimes overpaid.*

2. The second statement of Theorem 2 is equivalent to the following. For every player $q \in (U \cup V)$:

$$q \text{ is never paid } \iff q \text{ is not essential}$$

Some (counter-intuitive) corollaries

Corollary

In the assignment game, the set of essential players is non-empty.

Equivalently, since the set of essential players is the set of players in every maximum weight matching, we get that there are some set of vertices in the graph which are included in every maximum matching, which might sound counter-intuitive.

(Useful) Consequences of Vazirani's Theorem

1. Clearly the worth of the game is generated by teams that do play. Assume that (u, v) is such a team in an optimal assignment. Since $x_{uv} > 0$, by complementary slackness we get that $y_u + y_v = w_{uv}$, where y is a core imputation. Thus core imputations distribute the worth generated by a team among its players only.
2. Next we use Theorem 2 to get insights into degeneracy. Clearly, if an assignment game is non-degenerate, then every team and every player is either always matched or always unmatched in the set of maximum weight matchings in G, i.e., there are no viable teams or players.

Some (counter-intuitive) corollaries

Corollary

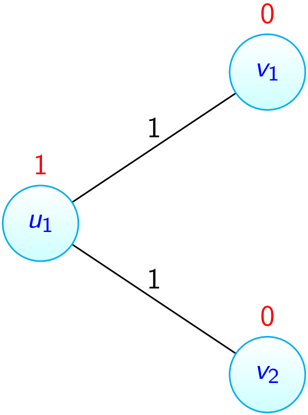
Imputations in the core of an assignment game treat viable and essential teams in the same way. Additionally, they treat viable and subpar players in the same way.

Insights into Negotiating Power of Agents

The purpose of this set of slides is a sanity check for what we just witnessed. Consider the below example. Using Shapley & Shubik theorem, the linear program which gives the assignment of the imputation is,

$$
\begin{aligned}
\min \sum_{i \in \mathcal{U}} u_i + \sum_{j \in \mathcal{V}} v_j & \\
u_i + v_j \geq w_{ij} & \qquad \forall (i,j) \in E, \\
u_i \geq 0 & \qquad \forall i \in \mathcal{U} \\
v_j \geq 0 & \qquad \forall j \in \mathcal{V}
\end{aligned}
$$

Example 1



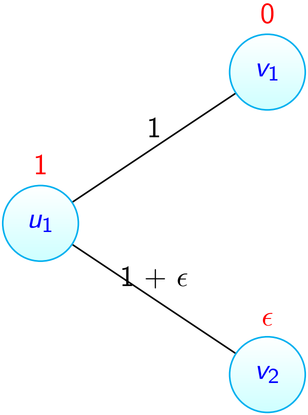
Assignment using LP

We have the LP,

$$\begin{aligned} \min u_1 + v_1 + v_2 \\ u_1 + v_1 \geq 1 \\ u_1 + v_2 \geq 1 \\ u_1, v_1, v_2 \geq 0 \end{aligned}$$

which gives the solution, $u_1 = 1, v_1 = 0, v_2 = 0$.

Example 2



Assignment using LP

We have the LP,

$$\begin{aligned} \min u_1 + v_1 + v_2 \\ u_1 + v_1 \geq 1 \\ u_1 + v_2 \geq 1 + \epsilon \\ u_1, v_1, v_2 \geq 0 \end{aligned}$$

with solution, $u_1 = 1, v_1 = 0, v_2 = \epsilon$.

How fair are these imputations?

Imputations in the core have a lot to do with the negotiating power of individuals and sub-coalitions. Let us argue that when the imputations given above are viewed from this angle, they are fair in that the profit allocated to an agent is consistent with their negotiating power, i.e., their worth.

1. In Example 1, whereas u has alternatives, v_1 and v_2 don't. As a result, u will squeeze out all profits from whoever she plays with, by threatening to partner with the other player. Therefore v_1 and v_2 have to be content with no rewards.
2. In Example 2, u can always threaten to match up with v_2. Therefore v_1 has to be content with a profit of ϵ only.

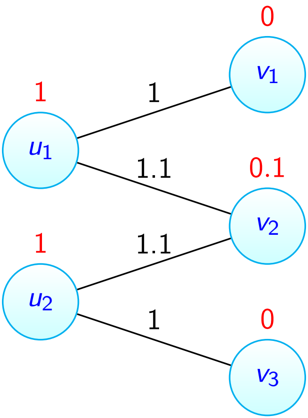
Secession from the coalition

In an arbitrary assignment game $G = (U, V, E), w$, by Vazirani's Theorem,

$$q \text{ is never paid } \iff q \text{ is not essential}$$

Thus core imputations reward only those agents who always play. This raises the following question.
Can't a non-essential player, say q, team up with another player, say p, and secede, by promising p almost all of the resulting profit? The answer is **No**, because the dual has the constraint $y_q + y_p \geq w_{qp}$. Therefore, if $y_q = 0, y_p \geq w_{qp}$, i.e., p will not gain by seceding together with q.

Example 3



Analysis

The worth of this assignment game, is clearly the size of the maximum matching, i.e. 2.1. Now, how do we distribute this worth among the players?

1. At first sight, v_2 looks like the dominant player, since he has two choices of partners, namely u_1 and u_2, and because teams involving him have the biggest earnings, namely 1.1 as opposed to 1.
2. Yet, the unique core imputation in the core awards $1, 1, 0, 0.1, 0$ to agents u_1, u_2, v_1, v_2, v_3, respectively.
3. So, why is v_2 allocated only 0.1? Can't he negotiate a higher profit, given his favorable circumstance? **No**.
4. The reason is that u_1 and u_2 are in an even stronger position than v_2, since both of them have a ready partner available, namely v_1 and v_3, respectively, with whom each can earn 1.

Vazirani's Theorem

Theorem

The following hold :

1. *For every team* $e \in E$*:*
$$e \text{ is always paid fairly } \iff e \text{ is viable or essential}$$
2. *For every player* $q \in (U \cup V)$*:*
$$q \text{ is paid sometimes } \iff q \text{ is essential}$$

Proof of Vazirani's Theorem

The proofs follow by applying complementary slackness conditions and strict complementarity to the primal LP (1) and dual LP (2). We will use Theorem 1 stating that the set of imputations in the core of the game is precisely the set of optimal solutions to the dual LP.

1. Let x and y be optimal solutions to LP (1) and LP (2), respectively. By the Complementary Slackness Theorem, for each $e = (u, v) \in E$,

$x_e(y_u + y_v - w_e) = 0$

Suppose e is viable or essential. Then there is an optimal solution to the primal, say x, under which it is matched, i.e., $x_e > 0$. Let y be an arbitrary optimal dual solution. Then, by the Complementary Slackness Theorem, $y_u + y_v = w_e$. Varying y over all optimal dual solutions, we get that e is always paid fairly. This proves the forward direction.

Proof of Vazirani's Theorem

For the reverse direction, we will use strict complementarity. It implies that corresponding to each team e, there is a pair of optimal primal and dual solutions x, y such that either $x_e = 0$ or $y_u + y_v = w_e$ but not both. For team e, assume that the right hand side of the first statement holds and that x, y is a pair of optimal solutions for which strict complementarity holds for e. Since $y_u + y_v = w_e$ it must be the case that $x_e > 0$. Now, since the polytope defined by the constraints of the primal LP (1) has integral optimal vertices, there is a maximum weight matching under which e is matched. Therefore e is viable or essential and the left hand side of the first statement holds.

Proof of Vazirani's Theorem

2. The proof is along the same lines and will be stated more succinctly. Again, let x and y be optimal solutions to LP (1) and LP (2), respectively. By the Complementary Slackness Theorem, for each
$q \in (U \cup V) : y_q(x(\delta(q)) - 1) = 0$
Suppose q is paid sometimes. Then, there is an imputation in the core, say y, such that $y_q > 0$. Therefore, for every primal optimal solution x, $x(\delta(q)) = 1$ and in every maximum weight matching in G, q is matched. Hence q is essential, proving the reverse direction. Strict complementarity implies that corresponding to each player q, there is a pair of optimal primal and dual solutions x, y such that either $y_q = 0$ or $x(\delta(q)) = 1$ but not both. Since we have already established that the second condition must be holding for x, we get that $y_q > 0$ and hence q is paid sometimes.